THEORY OF FUNCTIONS

AS APPLIED TO ENGINEERING PROBLEMS

EDITED BY

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TRANSLATOR'S PREFACE

IN converting this text into English the translator has followed the German of the authors as faithfully as possible. Whenever a question arose as to whether fidelity to the original or a graceful English version must rule, fidelity prevailed so long as it did not conflict with clarity. While not a slavishly literal translation, then, this is certainly not a loose one, for the nature of the text precludes looseness.

The notations of the original German edition have been retained almost in their entirety, the exception being the Gothic style letters for which bold face has been substituted in accordance with American typographical practice. In instances where technical terms peculiar to the German language and possessing no equivalents in English were encountered, or where there was any other difficulty in interpreting the author's meaning, the translator has introduced a comment designed to indicate clearly just what was originally meant.

American engineering owes a debt of gratitude to President Karl T. Compton of the Massachusetts Institute of Technology and to the late Mr. John R. Freeman for their support of this project to make available in English a valuable engineering treatise. The translator wishes to express his personal indebtedness to Philip Franklin, Associate Professor of Mathematics, and to Wilhelm Spannhake (who originally suggested the translation of this text), Visiting Professor of Applied Hydrodynamics at the Massachusetts Institute of Technology, for their helpfulness in resolving many questions which arose during the course of the translation.

THE TRANSLATOR.

Massachusetts Institute of Technology,
July, 1933.

PREFACE

MANY useful textbooks on the theory of functions are available in mathematical literature, the theory of functions representing one of the most important fields of mathematical science. The present book does not attempt to increase the number of those textbooks; rather, its goal is to be sought in the sub-title. In the development of exact sciences the methods of function theory play a steadily growing part. While the physicist — by virtue of his education — often comes in contact with the theory of functions, the engineer usually lacks a sufficient knowledge of these mathematical methods. There exist, it is true, a few procedures of applied mathematics specially adapted to technical problems, and these procedures often are based on the theorems of function theory. The development of theoretical engineering, however, makes it advantageous to free these special procedures from their symbols and to consider them — from a higher point of view — as a branch of general mathematics.

At the instigation of many practical engineers, the *Ausseninstitut* of the Berlin Institute of Technology, in its Winter Session of 1929–30, jointly with the Berlin Electrical Engineering Society [*Elektrotechnische Verein Berlin*] attempted to approach the above-mentioned goal by a simultaneous representation of this field of mathematics from the standpoint of pure mathematics as well as from that of applications. After eight introductory mathematical lectures by Mr. R. Rothe, a number of engineering applications were discussed. The limited time forced every lecturer to extreme brevity, so that the presented material can only mean a first introduction. The literature indicated in the text not only represents a bibliography of sources but is indispensable as far as extension and deepening of the material is concerned.

The material on applications was presented, partly because of the character of the audience — most of the hearers were members of the *Elektrotechnische Verein* — and partly by selection of the lecturers. Thus, one finds that a majority of the applications treated are to electrical engineering.

The editors are indebted to Messrs. Geheimrat Professor Dr. E. Orlich, and Oberingenieur Dr. C. Trettin for their stimulation, encouragement and assistance in preparing the lectures for publication. Thanks are also due to the book publisher, Julius Springer, for his coöperation.

THE EDITORS.

Berlin, October 1931.

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MATHEMATICAL FUNDAMENTALS
INTRODUCTION TO THEORY OF FUNCTIONS

BY R. ROTHE, BERLIN

A. COMPLEX QUANTITIES AND VARIABLES ANALYTIC FUNCTIONS

1. Explanation of Complex Quantities and their Interpretation as Vectors.

Any reader will probably know how to operate with complex quantities, i.e., with "quantities" of the form $a+i\,b$: in the same manner as with ordinary real quantities, merely putting $i^2=-1$. But there still remains much to be said concerning the nature of complex quantities and about their real meaning, particularly because of the confused conceptions about them which are so prevalent. The so-called "symbolic" method as used in alternating current analysis is, incidentally, nothing else but an operation with imaginary quantities.

In the following, the complex quantity $z=x+i\,y$ never means anything else but an assembled pair (x, y) of the real quantities x, y. Geometrically, they can be pictured in a plane (the "complex" z-plane) as abscissa and ordinate of a point (x, y), or z, in a rectangular cartesian system of coördinates, or else, they can be represented as the scalar components of a vector drawn from the origin to the point z, the vector itself being also denoted by z.

The $+$ sign in $z=x+i\,y$ shall mean that the vector z represents the *geometrical* sum (resultant) of the two vectors

$$x=(x, 0) \quad \text{and } i\,y=(0, y).$$

For $x=1$, $y=1$ the two unit vectors

$$1=(1, 0) \quad \text{and } i=(0, 1)$$

are obtained, at the same time revealing the real meaning and thus the conceptual existence of the so-called "imaginary" unit i : *i is the vector from the origin to the point 1 of the y-axis.*

As is well known, $\bar{z}=x-i\,y$ is called the conjugate vector to z and is symmetrical to z about the x-axis.

Instead of being represented by its components x, y the vector z can also be determined by its length or magnitude or modulus $r=|z|$ and its deviation or argument (or also "arcus" or "phase") $\varphi=\text{arc}\,(z)$, the latter given up to integer multiples of 2π (corresponding to a complete revolution). Thus, $|z|$ and arc (z) are simply the polar coördinates of the point z. From $x=r\cos\varphi$ and $y=r\sin\varphi$ (Fig. 1) it follows that

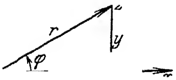


Fig. 1. The decomposition of a complex quantity.

$$z=x+i\,y=r(\cos\varphi+i\sin\varphi)=r\,e^{i\varphi}, \tag{1}$$

where

$$e^{i\varphi}=\cos\varphi+i\sin\varphi \tag{2}$$

signifies the vector of deviation φ and of unit magnitude. It is also evident that

$$|=r=\sqrt{x^2+y^2};\quad \text{arc}\,(z)=\cos^{-1}\frac{x}{r}=\sin^{-1}\frac{y}{r}=\tan^{-1}\frac{y}{r} \tag{3}$$

and that $|e^{i\varphi}|=\sqrt{\cos^2\varphi+\sin^2\varphi}=1$ and $e^{i\frac{\pi}{2}}=i$.

If z is a function of the real variable t, i.e. if x and y are functions of the real variable t, this means that a curve is represented in the z-plane: the locus of all points $z=x+i\,y$. If, for example, t stands for time and $\varphi=\varphi(t)$, $e^{i\varphi}$ represents a unit vector rotating about its origin. The procedure used in alternating current diagrams is based on the latter conception.

2. Notes Concerning Multiplication and Division of Complex Quantities.

The product of two complex quantities $z=z_1\cdot z_2$, by itself a complex quantity, is defined by $|z|=|z_1\cdot z_2|=|z_1|\cdot|z_2|$ and $\text{arc}\,(z)=\text{arc}\,(z_1z_2)=\text{arc}\,(z_1)+\text{arc}\,(z_2)$; the argument thus obeys the same law of addition as the function of the logarithm. Geometrically speaking, multiplication therefore consists in a change of deviation, i.e. rotation, with a simultaneous change of magnitude, i.e. extension or contraction.

Note especially the following relations:

$$|z^2|=|z|^2=z\bar{z}=x^2+y^2,\qquad \text{arc}\,(z^2)=2\,\text{arc}\,(z),$$

$$\text{arc}\,(e^{i\alpha}z)=\text{arc}\,e^{i\alpha}+\text{arc}\,(z)=\alpha+\text{arc}\,(z),\qquad |e^{i\alpha}z|=|e^{i\alpha}|\cdot|z|=|z|,$$

and thus $|i\,z|=|i|\,|z|=|z|$, as $|i|=1$; $\text{arc}(i\,z)=\text{arc}(i)+\text{arc}(z)=\frac{\pi}{2}+\text{arc}(z)$.

Multiplication of a vector by $e^{i\alpha}$ $(\alpha>0)$ therefore means pure rotation of the vector through the angle α in the positive sense of rotation.

Multiplication of a vector by i, in particular, means rotation through the angle $\frac{\pi}{2}$ in the positive sense.* Furthermore:

$$|i^2| = |i|^2 = 1; \qquad \text{arc } (i^2) = 2 \text{ arc } (i) = \pi,$$

and thus $\qquad i^2 = -1,$† $\qquad (-i)^2 = -1,$

$$|z^n| = |z|^n, \qquad \text{arc}(z^n) = n \text{ arc } (z) \text{ for } n > 0, \text{ and integer.} \tag{4}$$

Also note:

$$(e^{i\varphi})^n = e^{in\varphi}, \text{ i.e. } (\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i \sin n\varphi.$$

(Moivre's Theorem)

The operation $\quad z = \frac{z_1}{z_2} (z_2 \neq 0) \quad$ is explained as follows:

$$|z| = \qquad = \frac{|z_1|}{|z_2|}; \qquad \text{arc}(z) = \text{arc}\left(\frac{z_1}{z_2}\right) = \text{arc}(z_1) - \text{arc}(z_2).$$

By means of the notation

$$\frac{1}{z_1} = (z_1)^{-1} = \frac{\bar{z}}{|z|^2}$$

z^n is now also explained for negative integer n-values for all $z \neq 0$.

3. Extraction of Roots. Circle Division.

In order to explain the *roots* of complex quantities, find, with given z, all solutions w of the equation $w^n = z$ for $n > 0$ and integer. If we put

$$z = re^{i\varphi} \text{ (given)}, \qquad w = Re^{i\Phi} \text{ (required)},$$

it follows:

$$w^n = R^n \cdot (e^{i\Phi})^n = R^n \cdot e^{in\Phi} = z = r.e^{i\varphi}.$$

The equality of two vectors results in equality of their magnitudes and conformity of their arguments up to multiples of 2π. Therefore:

$$R^n = r,$$

$$\text{arc}(e^{in\Phi}) = \text{arc}(e^{i\varphi}) + k \cdot 2\pi \quad (k = 0, \pm 1, \pm 2, \ldots).$$

* The conception of $e^{i\alpha}$ as being a "symbol" for rotation through the angle α and of i as being a "symbol" for rotation through 90° in the positive sense is an inadequate one, for we always deal with quantities and vectors but never with "symbols."

† The equation $x^2 = -1$ thus has a solution which by all means makes sense, viz. $x = \pm i$, or $x = \pm\sqrt{-1}$. The only thing that has to be watched out for is the fact that -1 represents a certain vector, and one should not confuse the multiplication explained above with the multiplication of real quantities.

We thus have $R=\sqrt[n]{r}$, the uniquely determined real positive root being used for the absolute value of R, and

$$\Phi=\frac{\varphi}{n}+k\cdot\frac{2\pi}{n}\qquad(k=\pm0,\pm1,\pm2\ldots).$$

Now, the possible values of Φ repeat themselves as soon as k is increased or decreased by n. Hence, it is sufficient to take $k=0, 1, 2, \ldots (n-1)$. The equation under discussion therefore results in exactly n solutions:

$$w_k=Re^{i\Phi_k}=\sqrt[n]{r}\cdot e^{\left(i\frac{\varphi}{n}+k\frac{2\pi}{n}\right)},\qquad k=0, 1, 2, \ldots (n-1). \tag{5}$$

They are called n-th roots of $z: w_k=\sqrt[n]{z}$. Geometrically, the n corresponding points w_k are located on the circle with radius R about the origin as center and, what is more, these points are the division points of the periphery when the latter is divided into n equal parts, the angle 2π being divided n-times, starting at the point of intersection with the vector of argument $\frac{\varphi}{n}$ ("Circle division"). From the foregoing we conclude, for example, that $\sqrt{-1}$ is understood to represent the two vectors $+i$ and $-i$.

4. Complex Variables and Functions.

If x and y are variable, z is called a complex variable.

If z_0 represents a complex fixed quantity, $\lim z=z_0$ or $z\rightarrow z_0$ means that z eventually assumes only such values, for which $|z-z_0|<\epsilon$, no matter how small one may choose the positive real quantity ϵ.

If any kind of prescription coördinates values of some other complex variable w to those of z, w is called a complex function of z and is denoted by $w=f(z)$.

By means of the function $w=f(z)$, the points of the w-plane are referred to the points of the z-plane, or, in other words, the two planes are "mapped" one upon the other.

If z_0 and w_0 represent complex fixed quantities,

$$\lim_{z\rightarrow z_0} f(z)=w_0 \text{ or } w\rightarrow w_0 \text{ for } z\rightarrow z_0$$

means that, after an *arbitrary* choice of ϵ (real positive), a value $\delta(\epsilon)$ (also real positive) can be determined in such a way that $|f(z)-w_0|<\epsilon$ for all $|z-z_0|<\delta(\epsilon)$.

5. Continuity of Complex Functions.

A *real* function $f(x)$ of the real variable x is said to be continuous at the point x, if, for a given *arbitrary* positive ϵ, a positive function $\delta(\epsilon)$

can be determined in such a way that for *all* values h, for which $|h| < \delta(\epsilon)$, also

$$|f(x+h)-f(x)| \quad \text{becomes} < \epsilon.$$

Correspondingly, a complex function $f(z)$ is said to be continuous at the point z, if

$$|f(z+l)-f(z)| < \epsilon \text{ for all } |l| < \delta(\epsilon) \tag{6}$$

It must be remembered, however, that now $f(z+l)-f(z)$ is the difference of two vectors, also that l is a vector of the z-plane. The condition of continuity may also be put in the form:

$$\lim_{l \to 0} f(z+l) = f(z).$$

6. Application of Differential Calculus to Complex Functions.

If the difference-quotient of a real function of a real variable x can be represented in the following form:

$$\frac{f(x+h)-f(x)}{h} = f'(x) + \varphi(x, h),$$

i.e. can be resolved into two terms in such a way that the first is independent of h and for the second the relation holds: $h \to 0$, $\varphi(x, h) \to 0$, then, as is well-known, it is possible to differentiate $f(x)$, and $f'(x)$ is called the derivative of $f(x)$. Hence,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Dealing with real quantities h, being interpreted as a point of the real axis, may approach zero as closely as one may choose, starting from either positive or negative values.

The above definition of differentiation may be transferred to complex functions, as follows:

$$\lim_{l \to 0} \frac{f(z+l)-f(z)}{l} = f'(z); \tag{7}$$

or else

$$\frac{f(z+l)-f(z)}{l} = f'(z) + \varphi(z, l), \tag{8}$$

where

$$\lim_{l \to 0} \varphi(z, l) = 0.$$

In this case, however, l represents a vector, the end point of which can converge towards the origin following any arbitrary curve. The limit $f'(z)$ must be independent of the choice of this curve.

7. The Cauchy-Riemann Differential Equations.

Let

$$w=f(z)=u+iv=u(x, y)+iv(x, y),$$

where u and v are real functions of the real variables x and y.

Further, let $l=h+ik$. Now, the same result should be obtained for the derivative no matter which way we may choose for the convergence $l \to 0$. Thus, for example, the differentiation may be once performed letting

$$h=0, \quad k \to 0,$$

and another time letting

$$h \to 0; \quad k=0.$$

In the first case we obtain:

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\equiv\frac{\partial w}{\partial x} \tag{9}$$

In the second case, however, we have:

$$f'(z)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\equiv\frac{1}{i}\frac{\partial w}{\partial y} \tag{10}$$

Hence:

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Separating the real and imaginary parts, the Cauchy-Riemann differential equations follow:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}; \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \tag{11}$$

If the complex function $f(z)$ possesses a derivative at the point z, or for all values of z in a certain region, it is said to be *analytic* (regular) at these points.

It should be noted that the existence of $f'(z)$ may be deduced from that of equations (11), if only the four partial derivatives are continuous.

8. Equation of Potential.

By partial differentiation (if such is possible) of the Cauchy-Riemann equations it follows:

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{}{\partial y^2}\equiv\Delta u=0\\
&\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\equiv\Delta v=0
\end{aligned} \tag{12}$$

The real and the imaginary part of any analytic function satisfy the

same partial differential equation of second order, the equation of potential, the "general" solution with two arbitrary functions of which has the form:

$$u = F(x+iy) + G(x-iy).$$

9. Construction of Analytic Functions. Exponential Functions.

Now, let u and v be any two solutions of the equation of potential $\Delta u = 0$; these solutions are briefly called *harmonic* functions.

Then, generally, $u+iv$ is by no means an analytic function of $z = u+iy$. For a given harmonic function u the corresponding harmonic function v (in order to make $u+iv$ analytic) has to be determined in such a way as to assure the existence of the Cauchy-Riemann differential equations. Therefore,

$$dv \equiv \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

must be a total differential, and the integration gives

$$v = \int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) + C, \tag{13}$$

where C represents a real arbitrary constant.

Example: Let $u = e^x \cdot \cos y$; by formula (13) we have:

$$v = \int (e^x \sin y\, dx + e^x \cos y\, dy)$$

$$= \int d(e^x \sin y) = e^x \sin y + C.$$

Hence,

$$w = u + iv = e^x(\cos y + i \sin y) + C = e^x \cdot e^{iy} + C = f(z)$$

is an analytic function, thus possessing a derivative.
By equation (9)

$$f'(z) = e^x \cdot e^{iy} = w - C = f(z) - C.$$

Now, if we choose $C = 0$,

$$f'(z) = f(z) \text{ and } f(0) = 1$$

in agreement with the properties of the real exponential function e^x. For this reason $f(z) = e^x \cdot e^{iy}$ is said to be the exponential function in the complex plane and the following notation is used:

$$e^z = \exp(z) = e^x \cdot e^{iy} = e^{x+iy}. \tag{14}$$

For $x = 0$, we have:

$$e^{iy} = \exp(iy) = \cos y + i \sin y.* \tag{15}$$

* This substantiates what was said about $e^{i\alpha}$ in the first footnote on page 8; for, $e^{i\alpha}$ is not only a unit vector but also an exponential function.

By the way:

$$e^{z+k2\pi i}=e^{z} \qquad (k=0,\ \pm 1, \pm 2, \ldots)$$

i.e. the exponential function has the "fundamental" cycle $2\pi i$.

10. Further Examples of Analytic Functions. Elementary Functions.

Let

$$u=\ln|z|=\ln\sqrt{x^2+y^2}=\ln r.$$

Then we have:

$$\frac{\partial u}{\partial x}=u_x=+\frac{x}{r^2}; \qquad \frac{\partial^2 u}{\partial x^2}={}'' \quad \frac{y^2-x^2}{} .$$

$$\frac{\partial u}{\partial y}=u_y=+\frac{y}{r^2}; \qquad \frac{\partial^2 u}{\partial y^2}=u_{yy}=\frac{x^2-y^2}{r^4};$$

Hence, the equation of potential (12) $\Delta u=0$ is satisfied. Now, by equation (13),

$$=\int\left[\left(-\frac{y}{r^2}\right)dx+\left(\frac{x}{r^2}\right)dy\right]+C$$

$$=\int\frac{xdy-ydx}{r^2}+C=\int\frac{d\left(\frac{y}{x}\right)\cdot x^2}{r^2}+C=\int\frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}+C$$

$$=\int d\left(\tan^{-1}\frac{y}{x}\right)+C.$$

Thus,

$$v=\tan^{-1}\frac{y}{x}+C=\operatorname{arc}(z)+\text{const},$$

and a new analytic function has been obtained:

$$w=u+iv=\ln|z|+i\ \operatorname{arc}(z)+\text{const}.$$

For const. $=0$, this equation is called the logarithm of the complex argument z:

$$w=\log z=\ln|z|+i\ \operatorname{arc}(z)=f(z). \tag{16}$$

By equations (9) and (10) its derivative is:

$$\frac{dw}{dz}=\frac{d\log z}{dz}=f'(z)=\frac{\partial w}{\partial x}=-i\frac{\partial w}{\partial y}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y}$$

$$=\frac{x}{r^2}-i\frac{y}{r^2}=\frac{x-iy}{x^2+y^2}=\frac{1}{x+iy}=\frac{1}{z},$$

just as it would be in case of a real logarithm.

As arc (z) is only definite but for multiples of 2π, the logarithm is an infinitely many-valued function; its particular values belonging to the same argument z differ from each other by $k\cdot 2\pi i$ $(k=0, \pm 1, \pm 2, \ldots)$.

With the help of the logarithm a power of z with arbitrary exponent may be explained as follows:

$$z^n = e^{n\log z} \qquad (n \text{ arbitrary, also complex}).$$

Euler's Formula may serve as an example:

$$i^i = e^{i\log i} = e^{i\left(0+i\left(\frac{\pi}{2}+k2\pi\right)\right)} = e^{-\frac{\pi}{2}-k2\pi} \qquad (k=0,\pm 1,\pm 2,\ldots).$$

Furthermore, the integer rational function

$$g(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n,$$

where n is an integer and $n>0$, is now defined, as well as the fractional rational function

$$r(z) = \frac{a_0+a_1z+a_2z^2+\cdots+a_nz^n}{b_0+b_1z+b_2z^2+\cdots+b_mz^m} \qquad \begin{array}{l} n>0, \text{ integer} \\ m>0, \text{ integer} \end{array}$$

providing that the denominator does not vanish. The vanishing occurs at m points and $r(z)$ is not analytic at those points at which, by the way, $r'(z)$ is not existent. By means of the definition of z^n for arbitrary n-values $\sqrt{z}$, $\sqrt[3]{a_0+a_1z}$ etc. are explained and, finally, all "algebraic" functions, i.e. the solutions of algebraic equations, the coefficients of which are rational functions of z:

$$r_0(z)w^n + r_1(z)w^{n-1} + r_2(z)w^{n-2} + \cdots + r_{n-1}(z)w + r_n(z) = 0.$$

Example:

$$w = \sqrt{z^3 + \sqrt{a}z + b}.$$

The explicit representation by means of expressions containing roots is, however, not important for algebraic functions, and, what is more, not always possible (Abel Theorem).

There also exist transcendental, i.e. non-algebraic, functions. For example, neither a periodic function nor an infinite many-valued function can be algebraic. Transcendental functions thus are e^z and $\log z$, as well as the trigonometric functions derived from them:

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}), \qquad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz}), \tag{17}$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \tag{18}$$

Also, their inverse functions, the inverse trigonometric functions:

$$
\begin{aligned}
\sin^{-1} z &= \frac{1}{i}\log\left(iz \pm \sqrt{1-z^2}\right), \\
\cos^{-1} z &= \frac{1}{i}\log\left(z \pm \sqrt{z^2-1}\right), \\
\tan^{-1} z &= \frac{1}{i}\log\sqrt{\frac{1+iz}{1-iz}}, \\
\cot^{-1} z &= \frac{1}{i}\log\sqrt{\frac{iz-1}{iz+1}}
\end{aligned}
\tag{19}
$$

With respect to their applications, it seems worth while to study more carefully these functions in their form $u+iv$ or $Re^{i\Phi}$.*

11. One-Valued and Many-Valued Functions. Mapping of the z-plane upon the w-plane.

From the "elementary" functions mentioned above, the integer ones and the fractional rational functions, as well as the exponential and trigonometric functions are *one-valued* (uniform) functions, whereas the non-rational algebraic functions are *many-valued* (multiform, non-uniform), the logarithm and the arcus functions being even *infinitely many-valued.*

In case of the non-uniform functions, properly selecting a continuous succession of values, we may produce a "branch" of the many-valued function, which branch forms a one-valued function. This is done, for example, by preferring one of the two values of $\sqrt{z}$, or, in case of $\log z$, preferring that value, for which the modulus of periodicity is zero (main value of the logarithm):

$$\operatorname{Log} z = \ln|z| + i \operatorname{arc}(z),$$

where $0 \leqq \operatorname{arc}(z) < 2\pi$.

In the following discussion we shall at first consider one-valued functions only, including the above mentioned branches of many-valued functions.

It may be easily proved that the laws of differential calculus apply formally to the elementary analytic functions in the same manner as they apply to real functions. For example:

$$w = z^n = \exp(n \log z) = \exp\left(n \ln\sqrt{x^2+y^2} + in \tan^{-1}\frac{y}{x} + 2kn\pi i\right);$$

$$\frac{\partial w}{\partial x} = \exp(n \log z) \cdot n\,\frac{\partial \log z}{\partial x} = \exp(n \log z) \cdot n\frac{1}{z}$$

$$= nz^{n-1}.$$

* Compare F. Emde, "Sinus-und Tangensrelief in der Elektrotechnik." (Braunschweig, 1924.)

Thus, $(z^n)'=nz^{n-1}$, for all values of z, providing $n-1\geqq 0$, otherwise for $z\neq 0$ only.

Each function $w=f(z)$ of a complex variable z introduces a relation between the points of the z-plane, having coördinates x and y, and the points of the w-plane, having coördinates u and v. In case of one-valued functions, each point of the z-plane corresponds to *one* point of the w-plane at the most, but in case of many-valued functions, to a number of points.

The z-plane is said to be *mapped* upon the "*schlichte Ebene*" or the repeatedly overlapped w-plane. This mapping shows particular properties if $w=f(z)$ represents an analytic function. We shall return to this matter in the last section.

B. LINE-INTEGRALS IN THE REAL PLANE. RELATIONSHIP WITH POTENTIAL THEORY AND THEORY OF FLOW

1. Line-Integrals of Real Functions.

Let

$$p(x, y); \qquad q(x, y)$$

represent two real functions of the real variables x and y. We consider x and y to be one-valued functions of a new real variable t. Then,

$$x=x(t); \quad y=y(t) \text{ or } z=z(t)=x(t)+iy(t)$$

is the parameter representation of a curve. The curve is said to be "smooth," if $x'(t)$ and $y'(t)$ are continuous, i.e. the slope of the curve changes continuously.

The line-integral extended over a curve (C),

$$L=\int_{z_0\ (C)}^{z_1}(p(x, y)dx+q(x, y)dy)$$

is meant to be the following real definite integral:

$$L=\int_{t_0}^{t_1}[p(x(t), y(t))\cdot x'(t)+q(x(t).\ y(t))\cdot y'(t)]dt,$$

where t_0 and t_1 determine the limit-points $z_0=x_0+iy_0=x(t_0)+iy(t_0)=z(t_0)$ and $z_1=x_1+iy_1=x(t_1)+iy(t_1)=z(t_1)$.

The integral depends upon the functions p and q, the line section (C) and the limits z_0 and z_1. In order that the integral exists, it is satisfactory for the line section (C) to be smooth in a certain region.

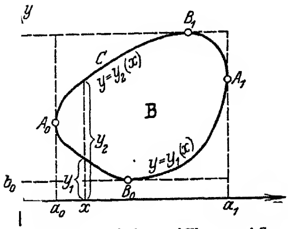


Fig. 2. Referring to the Integral Theorem of Gauss.

2. The Integral Theorem of Gauss.

Let (C) be a closed, continuous and smooth curve, which, as indicated in Fig. 2, is the boundary of a simply connected region $(\mathbf{B})$ of such property that any parallel to the y- or x-axis intersects the curve in two points at the most.* Let further p and q be two functions of x and y and let these functions be continuous on (C). Also, let $\frac{\partial q}{\partial x}$ and $\frac{\partial p}{\partial y}$ be existent and continuous inside of $(\mathbf{B})$.

Then, the following equation holds:

$$\iint\limits_{(\mathbf{B})} \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy = \oint\limits_{(C)} (p\,dx + q\,dy). \tag{20}$$

The double integral has to be taken over the region $(\mathbf{B})$ bounded by the curve (C), the line integral being taken along the boundary line (C) (circuit integral); this should be done in the mathematically positive sense, i.e. in such a way as to keep the inside of $(\mathbf{B})$ at the left.

Proof: We have (see Fig. 2):

$$\iint\limits_{(\mathbf{B})} \frac{\partial p}{\partial y}\,dx\,dy = \int\limits_{x=a_0}^{a_1}\int\limits_{y=y_1(x)}^{y_2(x)} \frac{\partial p}{\partial y}\,dx\,dy = \int\limits_{a_0}^{a_1}\left(\int\limits_{y_1(x)}^{y_2(x)} \frac{\partial p}{\partial y}\,dy\right)dx$$

$$= \int\limits_{a_0}^{a_1} [p(x, y_2(x)) - p(x, y_1(x))]\,dx = \int\limits_{a_0}^{a_1} p(x, y_2(x))\,dx - \int\limits_{a_0}^{a_1} p(x, y_1(x))\,dx$$

$$= \int\limits_{\substack{A_0\\(A_0, B_1, A_1)}}^{A_1} p\,dx - \int\limits_{\substack{A_0\\(A_0, B_0, A_1)}}^{A_1} p\,dx = -\int\limits_{\substack{A_1\\(A_1, B_1, A_0)}}^{A_0} p\,dx - \int\limits_{\substack{A_0\\(A_0, B_0, A_1)}}^{A_1} p\,dx,$$

* A bounded region is said to be simply connected, if any curve, connecting two arbitrary points of its boundary, splits it up into two separate regions.— The above mentioned additional property is not essential.

Thus:

$$\iint\limits_{(B)} \frac{\partial p}{\partial y}\, dx dy = -\oint\limits_{(C)} p dx.$$

A corresponding transformation of the surface integral

$$\iint\limits_{(B)} \frac{\partial q}{\partial x}\, dx dy$$

leads to $\oint\limits_{(C)} q dy$ and subtraction of both integrals results in the formula of the Gauss theorem.

3. Green's Theorem.

Substitute in Gauss' theorem:

$$p = -\varphi \frac{\partial \psi}{\partial y}; \qquad q = \varphi \frac{\partial \psi}{\partial x},$$

where φ has continuous first derivatives and ψ continuous second derivatives inside of (**B**) and along (C). Then we may write:

$$\iint\limits_{(B)} \varphi \Delta \psi dx dy + \iint\limits_{(B)} (\varphi_x \psi_x + \varphi_y \psi_y) dx dy = \oint\limits_{(C)} \varphi(\psi_x dy - \psi_y dx).$$

From Fig. 3 it is evident that

$$dx = ds \cdot \cos \vartheta = +ds \cdot \cos (\widehat{n, y}),$$

$$dy = dx \cdot \sin \vartheta = -ds \cdot \cos (\widehat{n, x}),$$

if ϑ designates the angle between the direction of the tangent of (C) (taken in the sense of rotation along the curve (C)) and the positive direction of the x-axis, and n designates the direction of the normal at the curve (C) (pointing towards the inside of (**B**)). Thus,

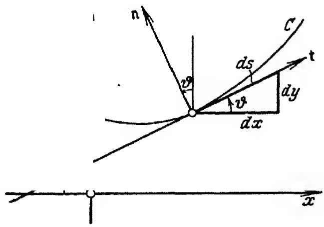


Fig. 3. Tangent, normal, element of arc.

$$\frac{\partial \psi}{\partial x} dy - \frac{\partial \psi}{\partial y} dx = -\left(\frac{\partial \psi}{\partial x} \cos(\widehat{n, x}) + \frac{\partial \psi}{\partial y} \cos(\widehat{n, y})\right) ds = -\frac{\partial \psi}{\partial n} \cdot ds,$$

where $\frac{\partial \psi}{\partial n}$ is the derivative of ψ taken along the direction of the normal.

Therefore, we may write:

$$\iint\limits_{(B)} \varphi \Delta \psi dx dy + \iint\limits_{(B)} \left(\frac{\partial \varphi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial y}\right) dx dy = -\oint\limits_{(C)} \varphi \frac{\partial \psi}{\partial n} ds. \qquad (21)$$

4. Condition of Integrability.

If the functions $p(x, y)$ and $q(x, y)$ satisfy the condition

$$\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y} \tag{22}$$

for any point within (B) and on the boundary (C), it follows from the integral theorem of Gauss (20), that

$$\oint(pdx+qdy)=0 \tag{23}$$

for the boundary (C) and any continuous closed and smooth curve within (B). If, on the other hand, the above circuit integral equals zero, the double integral must also be equal zero, i.e.

$$\left(\frac{\partial q}{\partial x}-\frac{\partial p}{\partial y}\right)dxdy=0,$$

the above expression holding for the region (B) and any component region of it. But this is only possible if the function under the integral sign vanishes. Condition (22) is therefore necessary and sufficient for the existence of equation (23).

In this case, the line integral

$$\int_{z_0\,(C)}^{z_1}(pdx+qdy)$$

does not depend any more on the path of integration (C), but only on the choice of the points z_0 and z_1. This may be illustrated by the following reasoning: for any two different paths (C) and (C') between z_0 and z_1 the equation

$$\int_{z_0\,(C)}^{z_1}(pdx+qdy)+\int_{z_1\,(C')}^{z_0}(pdx+qdy)\equiv\oint_{(C+C')}(pdx+qdy)=0$$

may be written and, hence,

$$\int_{z_0\,(C)}^{z_1}(pdx+qdy)=\int_{z_0\,(C')}^{z_1}(pdx+qdy).$$

With fixed z_0 the integral

$$\int_{z_0}^{z}(pdx+qdy)$$

thus becomes solely a function of the end-point z

$$\int^{z}(pdx+qdy)=F(x, y), \tag{24}$$

if condition (22) is satisfied, which condition, therefore, is called "condition of integrability." Thus, $pdx+qdy=dF(x,y)$, or else

$$p=\frac{\partial F}{\partial x}, \qquad q=\frac{\partial F}{\partial y}.$$

5. Illustration by Means of a Vector Field. Circulation and Potential.

Let $A=p+iq$ be a vector in the xy-plane and its components be $p(x, y)$ and $q(x, y)$. In the following discussion we shall interpret A as the velocity vector of a streaming fluid. The *work* of the vector along the curve (C) is — as also in mechanics — defined to be

$$\int_{z_0\,(C)}^{z}(pdx+qdy),$$

the scalar product of the vector A and the vectorial element of path $dx+idy$ appearing under the integral sign.

The work integral along a closed curve (boundary integral)

$$\oint_{(C)}(pdx+qdy),$$

is called *circulation* of the vector along (C) (or also eddy strength or strength of the vortex).

If the circulation along any closed line of the field is equal to zero, i.e.

$$\oint(pdx+qdy)=0$$

the field of the vector A is said to be *irrotational.* The necessary and sufficient condition for such a field is, from previous considerations,

$$\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}. \tag{25}$$

Then, we may write:

$$\int_{z_0}^{z}(pdx+qdy)=u(x, y),$$

$$p=\frac{\partial u}{\partial x}, \qquad q=\frac{\partial u}{\partial y}. \tag{26}$$

This function, $u=u(x, y)$, definite but for a constant, is called *potential* of the vector A. The vector A is said to be the gradient of u:

$$A=\frac{\partial u}{\partial x}+i\frac{\partial u}{\partial y}=\text{grad } u. \tag{27}$$

6. Divergence of a Vector Field.

The normal component A_n of the vector A, pointing towards the inside of (B), is (Fig. 4)

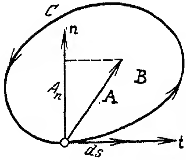


Fig. 4. Normal component of a vector.

$$A_n = p \cdot \cos \widehat{ux} + q \cdot \cos \widehat{uy} = -p \cdot \frac{dy}{ds} + q \cdot \frac{dx}{ds}.$$

Integrating, we get:

$$\oint A_n ds = \oint(-pdy + qdx) = \oint(qdx - pdy).$$

The integral at the left is the excess of the quantity of fluid flowing through (B) which leaves (B) over that which enters (B). This excess is called (depending upon the sign) *divergence or absorption*. If, however, the divergence vanished in each region (B) the field is said to be free of sources.

The necessary and sufficient condition for such a field is

$$\frac{\partial(-p)}{\partial x} = \frac{\partial q}{\partial y}, \tag{28}$$

or else

$$\operatorname{div} A \equiv \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} = 0.$$

7. Equation of Potential.

In an *irrotational* vector field having a potential u the latter, by equations (25) and (26), satisfies the differential equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{29}$$

One of the solutions of this equation is, by equation (12), the real part of $f(z)$,

$$u = \operatorname{Re} f(z),$$

where $f(z)$ represents an analytic function of the complex argument z. In a field *free of sources* we put, because of equation (28),

$$p = \frac{\partial v}{\partial y}; \qquad -q = \frac{\partial v}{\partial x}. \tag{30}$$

Because of equation (25), the *stream function* v also satisfies the equation of potential

$$\Delta v = 0.$$

8. The Irrotational and Sourceless Vector Field in a Plane.

In such a field equations (26) and (30) exist simultaneously, i.e.

$$p=\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}, \qquad q=\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x};$$

these equations will be recognized as the Cauchy-Riemann equations (11). Therefore, u and v are the components of an analytic function of z; v is said to be the *conjugate potential* of u. As, however, by equations (28) and (25),

$$\frac{\partial p}{\partial x}=\frac{\partial(-q)}{\partial y}; \qquad \frac{\partial p}{\partial y}=-\frac{\partial(-q)}{\partial x},$$

the *conjugate vector*

$$\tilde{\mathrm{A}}=p-iq$$

is also an analytic function of $z=x+iy$.

The curves of the xy-plane, having properties such that the direction of the tangent at any of its points falls together with the direction of the field vector at those points, are called *field lines*. Their slope, therefore, is

$$\frac{dy}{dx}=\frac{q}{p},$$

their differential equation

$$pdy-qdx=0 \tag{31}$$

The lines $u=$ constant are called *equipotential lines*. Their differential equation is

$$du\equiv pdx+qdy=0 \tag{32}$$

and, thus, their slope

$$\frac{dy}{dx}=-\frac{p}{q}.$$

From the above we follow: field lines cut equipotential lines at right angles.

The curves $v=$constant are called *stream lines* or *lines of flow;* their differential equation is

$$dv=-qdx+pdy=0,$$

in agreement with the field lines.

9. Example of an Irrotational and Sourceless Vector Field in a Plane.

Given the field vector:

$$A = \frac{x+iy}{x^2+y^2} = \frac{z}{|z|^2} = \frac{x}{r^2} + i\frac{y}{r^2} = \frac{1}{x-iy} = \frac{1}{\tilde{z}}.$$

Thus,

$$p = \frac{x}{r^2}, \qquad q = \frac{y}{r^2}.$$

The identical representation in polar coördinates is:

$$|A| = \left|\frac{1}{\tilde{z}}\right| = \frac{1}{|z|} = \frac{1}{r}; \qquad \text{arc }(A) = \tan^{-1}\frac{y}{x} = \text{arc }(z),$$

and thus

$$A = r^{-1}\cdot e^{i\varphi}, \qquad \tilde{A} = z^{-1} = r^{-1}\cdot e^{-i\varphi}.$$

The test by equations (25) and (28) shows that A is irrotational and free of sources, and, therefore, the two potentials can be computed:

$$u = \int_{z_0}^{z}(pdx+qdy) = \int_{z_0}^{z}\left(\frac{x}{r^2}dx + \frac{y}{r^2}dy\right) = \int_{z_0}^{z}\frac{xdx+ydy}{x^2+y^2}$$

$$= \frac{1}{2}\int_{z}^{z}\frac{d(x^2+y^2)}{x^2+y^2} = \ln\sqrt{x^2+y^2} + \text{const} = \ln|z| + \text{const},$$

$$v = \int_{z_0}^{z}\left(-\frac{y}{r^2}dx + \frac{x}{r^2}dy\right) = \int_{z_0}^{z}\frac{-ydx+xdy}{x^2+y^2} = \int_{z_0}^{z}\frac{x^2d\frac{y}{x}}{x^2+y^2} = \int_{z_0}^{z}\frac{d\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$$

$$= \tan^{-1}\left(\frac{y}{x}\right) + \text{const} = \text{arc }(z) + \text{const}.$$

Now,

$$w = u+iv = \ln(z) + i\text{ arc }(z) + \text{const} = \log z + \text{const}$$

is actually an *analytic* function of z, and

$$\frac{dw}{dz} = \frac{1}{z} = \frac{\partial u}{x} - i\frac{\partial u}{\partial y} = p - iq = \tilde{A}$$

is the conjugate vector and is also an analytic function of z, whereas A itself does not show this property.

10. Problem from Aeronautical Engineering.

Let us start from the analytic function

$$w=u+iv=\alpha\left(z+\frac{a^2}{z}\right)-i\beta \log z$$

which holds for real α, β and a. Let us, then, first determine the components of the conjugate vector

$$\bar{A}=\frac{dw}{dz}=\alpha\left(1-\frac{a^2}{z^2}\right)-i\beta\cdot\frac{1}{z}=p-iq$$

and from it, the vector $A=p+iq$ itself.

Let it be required to prepare a drawing of the stream lines $v=$ constant, the equipotential lines u = constant, and the "stagnation points" of the flow of fluid.*

The equipotential lines are the curves

$$u=\alpha\left(x+\frac{a^2x}{x^2+y^2}\right)+\beta\tan^{-1}\frac{y}{x}=\text{const},$$

the stream lines are the curves

$$v=\alpha\left(y-\frac{a^2y}{x^2+y^2}\right)-\beta\ln\sqrt{x^2+y^2}=\text{const}.$$

In particular, the latter include the circle $x^2+y^2=a^2$. It is on this circle that the *stagnation points*, i.e. the points at which $A=0$, are located.

As also $\bar{A}=\dfrac{dw}{dz}$ vanishes with A, the stagnation points can be computed from equation $\alpha z^2-i\beta z-\alpha a^2=0$. Thus, there exist two stagnation points (if $\alpha\neq 0$)

$$\left.\begin{matrix}z_1\\z_2\end{matrix}\right\}=i\frac{\beta}{2\alpha}\pm\sqrt{a^2-\frac{\beta^2}{4\alpha^2}}\cdot$$

If the square root is real, they are, because of $|z_1|=|z_2|=a$, both located on the above circle and are symmetrical to the y-axis.

C. INTEGRATIONS IN THE COMPLEX PLANE

1. Indefinite Integral in the Complex Plane.

If the relation

$$I'(z)=\frac{dI(z)}{dz}=f(z)$$

* See: Fuchs & Hopf, "Aerodynamik" (Berlin, 1922), p. 49.

exists between two analytic functions $I(z)$ and $f(z)$ within a region (**B**), i.e. if the function $f(z)$ is the derivative of $I(z)$ within the region (**B**), then (as also in the real plane) $I(z)$ is said to be an indefinite integral of $f(z)$.

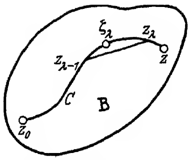


Fig. 5. Explanation of the definite integral in the complex plane.

The general solution of the above differential equation is

$$\int f(z)dz = I(z) + C,$$

where C represents an arbitrary (complex) constant of integration.

2. Definite Integral in the Complex Plane.

Let $f(z)$ be a complex (not necessarily analytic) function in the region (**B**) and let (C) be a smooth section of curve in (**B**), having the end points z_0 and z (see Fig. 5). Further, let

$$z_1,\ z_2,\ z_3 \quad ,\ z_{n-1}$$

be an arbitrary number of intermediate points on (C), z_n being chosen as point z. Then,

$$z_\lambda - z_{\lambda-1} = \Delta z_\lambda \qquad (\lambda = 1, 2, \ldots n)$$

represent chord vectors of the curve (C). Choose, finally, an additional point ζ_λ on curve (C), located between $z_{\lambda-1}$ and z_λ (this point may also coincide with $z_{\lambda-1}$ or with z_λ) and perform the line summation

$$\sum_{\lambda=1} f(\zeta_\lambda)\Delta z_\lambda = L_n. \tag{33}$$

If, splitting up the curve into smaller and smaller parts ($n \to \infty$, $|\Delta z_\lambda| \to 0$), this summation tends to approach a fixed limit-value, which is independent of the choice of the intermediate points and of the manner in which the division was performed, then, the above limit-value is called the *definite integral* of the complex function $f(z)$ taken along the path (C) and between the limits $z = z_0$ and $z = z$; it is denoted by:

$$\lim L_n = \int_{z_0\,(C)} f(z)dz. \tag{34}$$

The value of this integral depends on the function $f(z)$, but also on the limits of the integral and, generally, on the *form of the path* (C).

As in the case of real quantities, it may be shown that the integral, i.e. the lim L_n, exists, if $f(z)$ is continuous along the path of integration.

The relation between the definite integral and the indefinite one, the computation of the latter forming the main problem in calculus of

real integrals, shall now be established for analytic functions $f(z)$ by means of the following theorem.

3. The Principal Theorem of Function Theory.

Let $f(z)$ be analytic and one-valued throughout a simply connected region (**B**). Then, the following three assertions hold:

1. The definite integral

$$\int_{z_0\,(C)}^{z} f(z)dz$$

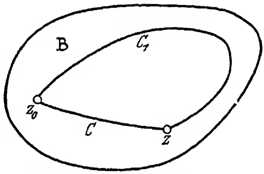


Fig. 6. Referring to the principal theorem of function theory.

is independent of the course of the path (C) and, therefore, is a function $F(z)$ of the upper limit z only.

2. $F(z)$ is also analytic in the region (**B**).

3. The derivative of $F(z)$ is identical with the function under the integral sign, i.e. with $f(z)$.

Taking for a moment the first assertion as having been proved and considering two arbitrary smooth paths (C) and (C_1) connecting the points z_0 and z and being completely located within (**B**) (Fig. 6), we can conclude the following:

$$\int_{z_0\,(C)} = \int_{z_0\,(C_1)} = -\int_{z\,(C_1)}$$

or else

$$\int_{z_0\,(C)} + \int_{z\,(C_1)} = 0$$

i.e.

$$\oint f(z)dz = 0. \tag{35}$$

Thus, the integral taken along an arbitrary closed path has, under the above assumptions, the value zero. In this form the first assertion is called the *integral theorem of Cauchy* or the *principal theorem of the theory of functions.*

4. Continuation. Proof of the Principal Theorem of Function Theory. (Assertion 1)

Returning to the line summation, equation (33),

$$L_n = \sum_{\lambda=1} f(\zeta_\lambda) \cdot \Delta z_\lambda$$

and choosing $\zeta_\lambda = z_\lambda$, we have:

$$z_\lambda = x_\lambda + iy_\lambda,$$
$$\Delta z_\lambda = \Delta x_\lambda + i\Delta y_\lambda,$$
$$f(z) = u(x, y) + iv(x, y),$$
$$f(\zeta_\lambda) = f(z_\lambda) = u(x_\lambda, y_\lambda) + iv(x_\lambda, y_\lambda).$$

Thus, we may write for the line summation:

$$L_n = \sum_{\lambda=1}^{n} [u(x_\lambda, y_\lambda) + iv(x_\lambda, y_\lambda)] \cdot [\Delta x_\lambda + i\Delta y_\lambda]$$
$$= \sum_{\lambda=1}^{n} [u(x_\lambda, y_\lambda) \cdot \Delta x_\lambda - v(x_\lambda, y_\lambda) \cdot \Delta y_\lambda]$$
$$+ i \sum_{\lambda=1}^{n} [v(x_\lambda, y_\lambda) \cdot \Delta x_\lambda + u(x_\lambda, y_\lambda) \cdot \Delta y_\lambda].$$

Now, going over to the limits, i.e. assuming

$$n \to \infty, \qquad \text{all } \Delta x_\lambda \to 0, \qquad \Delta y_\lambda \to 0,$$

L_n is transformed into the definite integral, equation (34), due to the fact that $f(z)$ is analytic in the total region and is, therefore, continuous also along the path of integration. The two summations on the right-hand side are transformed into real line integrals and we have:

$$\lim_{n\to\infty} L_n = \int_{z_0\ (C)}^{z} f(z)dz = \int_{x_0,\, y_0\ (C)}^{x,\, y} (udx - vdy) + i \int_{x_0,\, y_0\ (C)}^{x,\, y} (vdx + udy).$$

If we choose a closed path for (C), we obtain:

$$\oint_{(C)} f(z)dz = \oint (udx - vdy) + i \oint (vdx - udy).$$

By assumption, $f(z)$ is supposed to be analytic and, therefore, the Cauchy-Riemann differential equations must hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial} = -\frac{\partial v}{\partial x}.$$

But, by Gauss' integral theorem, the above relations represent the necessary and sufficient conditions for the vanishing of the two integrals. Thus, the proof* has been presented that

$$\oint f(z)dz = 0$$

* This proof, based upon Gauss' Integral Theorem, requires of course in addition to the above assumptions the ones used in the proof of Gauss' Integral Theorem itself, particularly the assumption that the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ be continuous. Other proofs of the principal theorem of function theory exist, proofs being free of the above assumptions.

or that $F(z)=\int_{z_0}^{z} f(z)dz$ is independent of the path

means the same.

5. Continuation. Proof of Assertions 2 and 3.

Decomposing,

$$F(z)=\int_{z_0\,(C)}^{z} f(z)dz = U(x,y)+iV(x,y),$$

we find from previous considerations:

$$U=\int_{z_0}^{z}(u\,dx-v\,dy),$$

$$V=\int_{z_0}^{z}(v\,dx+u\,dy).$$

By reason of the identity

$$U=\int_{z_0}^{z}\left(\frac{\partial U}{\partial x}dx+\frac{\partial U}{\partial y}dy\right),$$

the comparison of the two expressions for U gives

$$\frac{\partial U}{\partial x}=u,\qquad \frac{\partial U}{\partial y}=-v$$

and, in the same manner,

$$\frac{\partial V}{\partial x}=v,\qquad \frac{\partial V}{\partial y}=u.$$

Hence,

$$\frac{\partial U}{\partial x}=\frac{\partial V}{\partial y},\qquad \frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x},$$

which are the Cauchy-Riemann differential equations for the real and the imaginary components of $F(z)$.

Thus, $F(z)$ is analytic and contains a derivative $F'(z)$ at least in the same region (B) as does $f(z)$, which proves assertion 2.

Now that the existence of $F'(z)$ has been pointed out, we may write by equation (9)

$$F'(z)=\frac{d}{dz}F(z)=\frac{\partial}{\partial x}F(z)=\frac{\partial U}{\partial x}+i\frac{\partial V}{\partial x}$$

and from the preceding

$$F'(z)=u+iv=f(z),$$

thus proving assertion 3.

6. Multiply Connected Regions.

Cauchy's integral theorem had been deduced under the assumption that the closed curve was the boundary of a *simply* connected region, totally located in the definition-region (**B**) of the function $f(z)$. If, in addition, $f(z)$ should be analytic in all points of the boundary, Cauchy's integral theorem holds for the boundary as well. But, as we shall see, this theorem still holds if the enclosed region is *multiply* connected. Figure 7 indicates a region (**B**) of triple connection; this region requires three curves, connecting any two points of the boundary, in order to split it into two separate portions. Consider, for example, the cross-cuts AB and CD which do not split the region, only transforming it into a simply connected region; it should be noted, however, that the cross-cuts AB and CD should never be intersected.

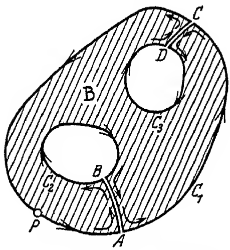


Fig. 7. Triply connected region showing cross-cuts.

To deduce Cauchy's theorem for multiply connected regions (**B**), we transform this region into a simply connected one, introducing appropriate cross-cuts, such as AB and CD. Then, starting from an arbitrary point P of the boundary, we proceed along the curve in the direction as indicated by arrows in Fig. 7, finally returning to P. The chosen direction happens to be the mathematically positive one, the inside of the region always remaining at the left side. Note that the cross-cuts AB and CD, connecting the separated boundaries (C_1), (C_2) and (C_3), should be followed in both directions: i.e. entering the region from the outside boundary and returning to the outside boundary. If $f(z)$ represents an analytic function within (**B**) and on all boundaries (C_1), (C_2) and (C_3) this function must be analytic on AB and CD as well. Thus, the integral $\oint f(z)dz$ extended over the total boundary of the opened (and, therefore, simply connected) region is equal to zero. This integral is composed of a sum of several component integrals, the ones extended over AB and BA, CD and DC cancelling out. We thus may write

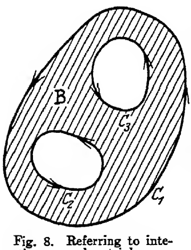


Fig. 8. Referring to integration around a triply connected region.

$$\oint_{(C_1)}+\oint_{(C_2)}+\oint_{(C_3)} f(z)dz=0,$$

where (C_1) is taken in the mathematically positive sense, (C_2) and (C_3)

in the negative one. If all boundaries are followed in the same (positive) sense, as indicated in Fig. 8, we have:

$$\oint_{(C_1)} f(z)dz = \oint_{(C_2)} f(z)dz + \oint_{(C_3)} f(z)dz. \tag{36}$$

This formula is important because it holds even if $f(z)$ is not analytic inside of the portions enclosed by (C_2) and (C_3). The case of a doubly connected region is quite worthy of consideration (Fig. 9). Here

$$\oint_{(C)} f(z)dz = \oint_{(K)} f(z)dz; \tag{37}$$

i.e. the closed path of integration (C) may be substituted by any other path (K), completely located inside of (C), if only $f(z)$ is analytic in the annular portion between (C) and (K) as well as on (K) itself; $f(z)$ does not have to be analytic inside of (K).

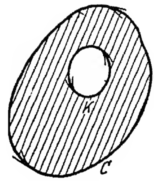


Fig. 9. Doubly connected region.

7. Examples Illustrating the Principal Theorem of Theory of Functions.

(a) Let it be required to compute the following integral:

$$\oint_{(C)} \frac{dz}{z}.$$

The function $f(z) = \dfrac{1}{z}$ is analytic for any value of z, except for $z=0$. If, therefore, the simply closed curve (C) does not surround the origin, the integral has the value zero. If, on the other hand, (C) does surround the origin, let us enclose the origin by a circle (K) of sufficiently small radius ρ such that (K) be completely located within (C) (Fig. 10). Then, by equation (37)

$$\oint_{(C)} \frac{dz}{z} = \oint_{(K)} \frac{dz}{z}.$$

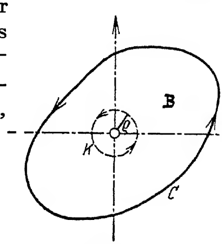


Fig. 10. Path of integration surrounds the origin.

Introducing

$$z = \rho \cdot e^{\varphi}, \qquad \rho = |z|; \qquad \varphi = \text{arc } (z),$$

we have for the circle (K)

$$\rho=\text{const}, \ \varphi=0 \ldots 2\pi,$$

and thus:

$$dz=\rho\cdot e^{i\varphi}\cdot i d\varphi, \qquad \frac{dz}{z}=i\cdot d\varphi,$$

hence

$$\oint\frac{dz}{z}=i\int_0^{2\pi}d\varphi=2\pi i.$$

The result is:

$$\oint_{(C)}\frac{dz}{z}=\begin{cases}0\\2\pi i\end{cases}, \text{ depending upon whether } (C)\begin{cases}\text{does not surround}\\\text{does surround}\end{cases}\text{the origin.} \tag{38}$$

(b) Let it be required to compute

$$\oint_{(C)}(z-z_0)^m dz, \text{ where } m \text{ is an integer.}$$

If $m\geqq 0, f(z)=(z-z_0)^m$ is analytic for any value of z, the integral then having the value zero. But if $m<0$, $f(z)$ is not any longer analytic for $z=z_0$. If the path of integration (C) does not surround the point z_0 the value of the integral is still zero. But if (C) does surround the point z_0, let us enclose z_0 by a circle (K) of sufficiently small radius ρ, so that (K) comes to lie completely within (C) (Fig. 11). Then, by equation (37):

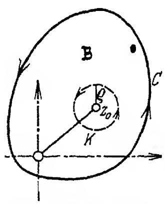


Fig. 11. Path of integration surrounds the point z_0.

$$\oint_{(C)}(z-z_0)^m dz=\oint_{(K)}(z-z_0)^m dz.$$

With

$$z-z_0=\rho\cdot e^{i\varphi}, \qquad \rho=\text{const}; \qquad dz=\rho e^{i\varphi}\cdot i d\varphi$$

the integral along the circle (K) becomes:

$$\oint_{(K)}=\int_0^{2\pi}\rho^m\cdot e^{im\varphi}\cdot\rho e^{i\varphi}\cdot i d\varphi=i\rho^{m+1}\cdot\int_0^{2\pi}e^{i(m+1)\varphi}d\varphi=i\rho^{m+1}\cdot\left[\frac{e^{i(m+1)\varphi}}{i(m+1)}\right]_0^{2\pi}$$

$$=\frac{\rho^{m+1}}{m+1}\cdot[e^{i(m+1)2\pi}-e^0]=0, \text{ if } m+1\neq 0.$$

For $m=-1$, the value of the integral is $2\pi i$. Thus:

$$\oint_{(C)}(z-z_0)^m dz=\begin{cases}0 & \text{generally,}\\2\pi i, & \text{if } m+1=0 \text{ and } (C) \text{ surrounds the point } z_0.\end{cases} \tag{39}$$

8. Estimation of the Integral Value.

Let $f(z)$ be an analytic function on the curve (C) (which may be rectified) and a restricted function (with respect to its value) along the curve section from z_0 to z, i.e.

$$|f(z)| \leqq M,$$

where M represents a fixed real quantity. Then, the following estimation of the integral along (C) exists:

$$\left| \int_{z_0}^{z} f(z) dz \right| \leqq \text{M} \cdot l, \tag{40}$$

where l is the length of arc of the curve section (C) from z_0 to z.

Proof: Substituting the absolute values in the line summation of equation (33) and for $|f(\zeta_\lambda)|$ the upper limit M, we have:

$$\left| \int_{z_0}^{z} f(z)\, dz \right| < M \cdot \lim \sum_{\lambda=1}^{n} |\Delta z_\lambda|.$$

Now,

$$\lim \sum_{\lambda=1}^{n} |\Delta z_\lambda| = \int_{z_0}^{z} |dz| = \int_{z_0}^{z} \sqrt{dx^2+dy^2} = l,$$

which proves equation (40).

9. Remarks Concerning the Principal Theorem of Theory of Functions.

(a) Under article 14 we shall prove the following: if $f(z)$ is continuous in all points of a simply connected region (**B**) and if

$$\oint_{(C)} f(z) dz = 0$$

for any closed curve (C) in (**B**), then $f(z)$ is analytic in (**B**). This theorem (of Morera) therefore represents the reversal of Cauchy's integral theorem.

(b) There are, of course, cases in which

$$\oint_{(C} f(z) dz = 0$$

without $f(z)$ being analytic inside of (C). In the following case, for example, a case encountered frequently enough, we may conclude from the properties of $f(z)$ along the path of integration (C) that the integral must vanish: first, if $f(z)$ is continuous along (C) — otherwise the very existence of the integral being questionable — second, if $f(z)$ happens to be the derivative of another function $F(z)$, which, in turn, is analytic and

continuous along (C). For, putting $f(z)=F'(z)$ or $f(z)dz=dF(z)$ and assuming on (C) two arbitrary points z_0 and z_1, we obtain:

$$\oint_{(C)} f(z)dz=\int_{z_0}^{z_1} dF(z)+\int_{z_1}^{z_0} dF(z)=F(z_1)-F(z_0)+F(z_0)-F(z_1)=0.$$

Compare also the example (b) of Art. 7, putting $m \leqq -2$.

(c) Sometimes we may compute the integral

$$\int_{z_0\,(C)}^{z} f(z)dz$$

directly from the conception of the limit-value of a line summation (equation 33). If, for example, $f(z)=1$, we have

$$\int_{z_0}^{z} dz=\lim \sum_{\lambda=1}^{n} (z_\lambda-z_{\lambda-1})=\lim\ (z-z_0)=z-z_0, \tag{41}$$

which is independent of the path of integration. Therefore,

$$\oint dz=0 \tag{42}$$

Or, for $f(z)=z$, substitute first $\zeta_\lambda=z_\lambda$ and then $\zeta_\lambda=z_{\lambda-1}$, obtaining the line summations

$$L_n=\sum_{\lambda=1}^{n} z_\lambda(z_\lambda-z_{\lambda-1})$$

and

$$L'_n=\sum_{\lambda=1}^{n} z_{\lambda-1}(z_\lambda-z_{\lambda-1})$$

the sum of which is

$$L_n+L'_n=\sum_{\lambda=1}^{n} (z_\lambda{}^2-z_\lambda z_{\lambda-1}+z_{\lambda-1}z_\lambda-z^2{}_{\lambda-1})=z^2-z_0{}^2.$$

Therefore,

$$\lim L_n+\lim L'_n=\int_{z_0}^{z} zdz=z^2-z_0{}^2 \tag{43}$$

is also independent of the path of integration, yielding

$$\oint zdz=0. \tag{44}$$

10. Cauchy's Integral Formula.

Again, let $f(z)$ represent an analytic function within the region **(B)** and let us now consider the function

$$\frac{f(z)}{z-z_0},$$

where z_0 indicates a point located inside of (B). This function is not analytic in (B). But if we exclude the point z_0 from the region (B) by means of a circle (K) of sufficiently small radius ρ (compare Fig. 11), we have by equation (37):

$$\oint_{(C)} \frac{f(z)dz}{z-z_0} = \oint_{(K)} \frac{f(z)dz}{z-z_0}.$$

Introducing polar coördinates, we obtain for the integral along (K):

$$z=\rho e^{i\varphi}+z_0; \qquad dz=\rho e^{i\varphi}\cdot i d\varphi; \qquad \frac{dz}{z-z_0}=i d\varphi;$$

$$\oint_{(C)} \frac{f(z)dz}{z-z_0} = \int_0^{2\pi} f(\rho e^{i\varphi}+z_0)\cdot i d\varphi. \tag{45}$$

As $f(z)$ was assumed to be analytic, the condition of continuity

$$|f(z_0+\rho e^{i\varphi})-f(z_0)|<\epsilon$$

holds if

$$|\rho e^{i\varphi}|<\eta(\epsilon), \text{ i.e. } |\rho|<\eta(\epsilon)$$

and if ϵ denotes an arbitrarily small positive number, $\eta(\epsilon)$, however, denoting a positive number determined by ϵ. This condition may also be written in the form:

$$f(z_0+\rho e^{i\varphi})=f(z_0)+\vartheta,$$

where ϑ signifies a complex quantity with the property $|\vartheta|<\epsilon$. Now the integral of equation (45) may be transformed as follows:

$$\oint_{(C)} \frac{f(z)dz}{z-z_0} = i\int_{\varphi=0}^{2\pi} f(z_0)d\varphi + i\int_{\varphi=0}^{2\pi} \vartheta d\varphi = 2\pi i f(z_0) + i\int_{\varphi=0}^{2\pi} \vartheta d\varphi.$$

The second integral may be estimated:

$$\left|\int_{\varphi=0}^{2\pi} \vartheta d\varphi\right| < \epsilon \int_{\varphi=0}^{2\pi} d\varphi = \epsilon\cdot 2\pi.$$

We thus obtain the result:

$$\left|\oint_{(C)} \frac{f(z)\cdot dz}{z-z_0} - 2\pi i f(z_0)\right| < \epsilon\cdot 2\pi.$$

But as we may choose ϵ arbitrarily small, we also may write:

$$\oint \frac{f(z)\cdot dz}{z-z_0} = 2\pi i\cdot f(z_0), \tag{46}$$

or else, using a different notation:

$$f(z)=\frac{1}{2\pi i}\oint\limits_{(C)}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{47}$$

This is *Cauchy's integral formula;* it gives us instructions how to compute the value of a function $f(z)$ inside a simply connected region (**B**) from the values of the function at the boundary (C). This formula, likewise, holds for multiply-connected regions, if only all the individual boundaries are circuited in the mathematically positive sense (the inside to the left).

If we apply formula (47) to a circle with the center z the substitutions $\zeta-z=re^{i\varphi}$ and $d\zeta=(\zeta-z)i\,d\varphi$ have to be made and we obtain

$$f(z)=\frac{1}{2\pi}\int\limits_{\varphi=0}^{2\pi}f(\zeta)\,d\varphi, \tag{48}$$

indicating that the functional value at the center of a circle is equal to the arithmetic mean value of its boundary values. If M denotes the maximum of the boundary values $|f(\zeta)|$, we get, by the formula of estimation, equation (40):

$$|f(z)|\leqq M. \tag{49}$$

From the above relation we conclude that the maximum of the absolute value of an analytic function can not be situated inside of a circular region, unless $f(z)$ itself possesses the fixed value M at all points.

11. Poisson's Integral.

Referring to Cauchy's integral formula, let us choose as boundary curve a circle of radius R located within the region (**B**) of function $f(z)$, and let the center of this circle be taken as the origin of the z-plane; this is always possible by putting $z-c$ instead of z, where c indicates the complex constant pertaining to the center. Let $z=r\cdot e^{i\varphi}$ be an inside point of this circle $(r<R)$. We then have, by Cauchy's integral formula, equation (47), introducing $\zeta=R\cdot e^{i\vartheta}$,

$$f(r\cdot e^{i\varphi})=\frac{1}{2\pi}\int\limits_{\vartheta=0}^{2\pi}\frac{f(R\,e^{i\vartheta})\cdot Re^{i\vartheta}}{Re^{i\vartheta}-re^{i\varphi}}\,d\vartheta. \tag{50}$$

Let further $z_1=r_1\cdot e^{i\varphi_1}$ represent a point in (**B**), outside of the circle $(r_1>R)$. Then, the expression

$$\frac{f(z)}{z-z_1}$$

is analytic inside of the circle and the curve-linear integral of this function taken along the circle of radius R must vanish:

$$\frac{1}{2\pi}\int_{\vartheta=0}^{2\pi}\frac{f(Re^{i\vartheta})\cdot Re^{i\vartheta}}{Re^{i\vartheta}-r_1e^{i\varphi_1}}\,d\vartheta=0. \tag{51}$$

In particular, let us choose z_1 such as to make

$$\text{arc } (z_1)=\text{arc } (z), \quad r_1=\frac{R^2}{r}.$$

With this choice the point z_1 is obtained from z by virtue of a transformation using reciprocal radii, the center of which is the origin (which latter is the center of the circle of radius R). Figure 12 indicates the construction of z_1 with given z.

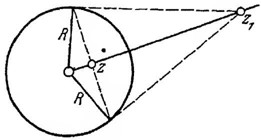


Fig. 12. Reciprocal radii

Subtracting integral (51) from that of equation (50), we may, leaving out a few simple steps of the calculation, sum up the result in the following manner:

$$f(r\cdot e^{i\varphi})=\frac{1}{2\pi}\int_{\vartheta=0}^{2\pi}f(Re^{i\vartheta})\,\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\vartheta)+r^2}\,d\vartheta. \tag{52}$$

Now, resolve $f(z)=w$ into the real component u and the imaginary component v; then, a formula of the form

$$u(r,\varphi)=\frac{1}{2\pi}\int_{\vartheta=0}^{2\pi}u(R,\vartheta)\cdot\frac{R^2-r^2}{R^2-2Rr\cos(\varphi-\vartheta)+r^2}\,d\vartheta \tag{53}$$

applies to both of them. This formula is called *Poisson's integral.* As u satisfies the equation of potential $\Delta u=0$, Poisson's integral solves, for a circular region, the so-called first problem of boundary-values of the potential theory. Of course, the potential for any arbitrary point may then be computed from the boundary values. For the center $(r=0)$ in particular, we have

$$u(0,0)=\frac{1}{2\pi}\int_{\vartheta=0}^{2\pi}u(R,\vartheta)\cdot d\vartheta, \tag{54}$$

u (0, 0) thus being found to represent the mean value of u along the boundary for the polar angle ϑ as the argument. From the above, the further conclusion may be drawn that $u(r,\varphi)$ which, being a continuous function, must contain a maximum and a minimum in any bounded region has to assume this largest and smallest value on the boundary itself.

12. Derivatives of an Analytic Function.

A function, analytic within a region (**B**) and on the boundary (C), contains at any inside point of (**B**) derivatives of arbitrary order, these derivatives being all analytic in (**B**).

To prove the above statement let us form the difference quotient

$$\frac{f(z_0+h)-f(z_0)}{h},$$

where z_0 as well as z_0+h are assumed to lie in the region (**B**). By means of Cauchy's integral formula (46) we may put this expression in the form:

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2\pi i h}\cdot\left\{\oint_{(C)}\frac{f(z)dz}{z-z_0-h}-\oint\frac{f(z)dz}{z-z_0}\right.\\ &= \frac{1}{2\pi i}\oint\frac{f(z)dz}{(z-z_0)(z-z_0-h)}\\ &= \frac{1}{2\pi i}\cdot\left\{\oint\frac{f(z)dz}{(z-z_0)^2}+h\cdot\oint\frac{f(z)dz}{(z-z_0)^2(z-z_0-h)}\right\}.\end{aligned}$$

Using an appropriate limitation for $|h|$ we have

$$\left|\frac{f(z)}{(z-z_0)^2(z-z_0-h)}\right|\leqq M;$$

we may, therefore, estimate by equation (40):

$$\left|\frac{f(z_0+h)-f(z_0)}{h}-\frac{1}{2\pi i}\oint_{(C)}\frac{f(z)dz}{(z-z_0)^2}\right|\leqq M\cdot l\frac{|h|}{2\pi},$$

where l indicates the length of arc of the path of integration (C). Now, going over to the limit $h\rightarrow 0$, we get

$$\lim_{h\rightarrow 0}\frac{f(z_0+h)-f(z_0)}{h}\equiv f'(z_0)=\frac{1}{2\pi i}\oint_{(C)}\frac{f(z)dz}{(z-z_0)^2},$$

or, using a different notation:

$$f'(z)=\frac{1}{2\pi i}\oint_{(C)}\frac{f(\zeta)d\zeta}{(\zeta-z)^2}. \tag{55}$$

By comparison with integral formula (47) for $f(z)$, it is realized that we may formally perform the differentiation with respect to z under the

integral sign. Repeating this conclusion, we find that we may continue to differentiate under the integral sign and obtain:

$$f''(z)=\frac{2}{2\pi i}\oint\frac{f(\zeta)d\zeta}{(\zeta-z)^3}, \tag{56}$$

$$f^{(n)}(z)=\frac{n!}{2\pi i}\oint\frac{f(\zeta)d\zeta}{(\zeta-z)^{n+1}}. \tag{57}$$

Formula (57) represents the n-th derivative of an analytic function in the form of a definite integral, the function $f(z)$ itself appearing under this integral.

13. Continuation. Examples.

From the conclusions of the preceding article we may evidently state that *an analytic function $f(z)$ (in a certain region) possesses derivatives of arbitrarily high order in that region and that all these derivatives are likewise analytic functions.*

This is a puzzling result in as much as from the fact alone that $f(z)$ may be differentiated, it necessarily follows that $f(z)$ also possesses derivatives of any order. As is well known, this is not at all the case when dealing with reals. We cannot even assert that, if $f(x)$ may be differentiated, the derivative $f'(x)$ is continuous, not to speak of the assertion that $f'(x)$ may be differentiated.

Let us now apply Cauchy's formulas (47), (55), (56) and (57) to a few simple functions.

(a) For $f(z)=1$, equation (47) directly furnishes

$$\oint\frac{d\zeta}{\zeta-z}\quad 2\pi i$$

in agreement with equation (39); and, as every derivative of $f(z)$ vanishes, equation (57) yields:

$$\oint\frac{d\zeta}{(\zeta-z)^{n+1}}\quad 0 \qquad (n\geq 1)$$

(b) Let $f(z)=z$; then, by equation (47), we have:

$$2\pi iz=\oint\frac{\zeta d\zeta}{\zeta-z}. \tag{58}$$

It is quite worth while to prove this formula by direct computation. To do that substitute curve (C) by a circle (K) of radius ρ and center z,

which circle is totally surrounded by (C). (See Fig. 11.) The integral of equation (58) is, by equation (37), equal to the integral taken along the circle. If we put $\zeta=z+\rho e^{i\varphi}$, φ is the variable of integration along the circle (K), and we finally obtain:

$$\oint_{(C)}\frac{\zeta d\zeta}{\zeta-z}=\oint_{(K)}\frac{\zeta d\zeta}{\zeta-z}=\int_{\varphi=0}^{2\pi}(z+\rho e^{i\varphi})id\varphi=2\pi iz+\rho i\int_{\varphi=0}^{2\pi}e^{i\varphi}d\varphi=2\pi iz,$$

as the last integral vanishes. We thus have the same result as previously.

(c) Again, let $f(z)=z$. Application of formulas (55) and (57) directly yields:

$$2\pi i=\oint\frac{\zeta d\zeta}{(\zeta-z)^2},\qquad 0=\oint\frac{\zeta d\zeta}{(\zeta-z)^n}\qquad (n>2,\text{ integer}). \tag{59}$$

At this point, as a matter of exercise, the reader should attempt to compute these integrals by means of direct calculations.

(d) Let $f(z)=z^m$ (m arbitrary). If $m<0$, the origin should lie outside of the region enclosed by (C). We have now

$$f^{(n)}(z)=m(m-1)(m-2)\cdots(m-n+1)z^{m-n},$$

and by equation (57):

$$2\pi i\cdot m(m-1)(m-2)\cdots(m-n+1)z^{m-n}=n!\oint\frac{\zeta^m d\zeta}{(\zeta-z)^{n+1}}$$

or, dividing by $n!$ and introducing the binomial coefficient, we finally get:

$$2\pi i\binom{m}{n}z^{m-n}=\oint\frac{\zeta^m d\zeta}{(\zeta-z)^{n+1}}\qquad \begin{matrix}(m\text{ arbitrary;}\\ n\geq 0,\text{ integer}).\end{matrix} \tag{60}$$

(e) Finally, let $f(z)=e^z$. Then $f^{(n)}(z)$ is also equal e^z and we have by equation (57)

$$2\pi ie^z=n!\oint\frac{e^\zeta d\zeta}{(\zeta-z)^{n+1}}. \tag{61}$$

The curve of integration may be arbitrarily located in the z-plane as the exponential function is analytic in the total plane. Again, let us substitute the curve of integration by a circle (K) about the point z as its center, having a sufficiently small radius. For $\zeta=z+\rho e^{i\varphi}$, equation (61) becomes:

$$2\pi ie^z=n!\frac{e^z}{\rho^n}\int_{\varphi=0}^{2\pi}\exp(\rho e^{i\varphi})\cdot e^{-ni\varphi}id\varphi,$$

and hence

$$2\pi\rho^n = n!\int_{\varphi=0}^{2\pi} \exp\left(\rho\cos\varphi + i(\rho\sin\varphi - n\varphi)\right)d\varphi.$$

Splitting the real part from the imaginary one:

$$2\pi\frac{\rho^n}{n!} = \int_{\varphi=0}^{2\pi} e^{\rho\cos\varphi}\cos(\rho\sin\varphi - n\varphi)d\varphi, \tag{62}$$

$$0 = \int_{\varphi=0}^{2\pi} e^{\rho\cos\varphi}\sin(\rho\sin\varphi - n\varphi)d\varphi. \tag{63}$$

These integrals surely are more cumbersome to compute if one only uses the methods of integral calculus in the real plane.

14. Proof of Morera's Theorem.

As was already mentioned under Art. 9, this theorem is the reversal of Cauchy's integral theorem and may be stated as follows: if $f(z)$ is wholly continuous in a simply connected region (**B**), and if

$$\oint_{(C)} f(z)dz = 0$$

for any closed curve (C) in (**B**), then $f(z)$ is wholly analytic within (**B**).

Proof: As any closed integral vanishes within (**B**), the line-integral

$$F(z) = \int_{\zeta=z_0}^{z} f(\zeta)d\zeta$$

is independent of the path and is a function of the end point z alone. Now, if $z+h$ (as also z) is situated within (**B**), we have:

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int^{z+h} f(\zeta)d\zeta.$$

But, from the assumed continuity of the function $f(z)$ it follows:

$$f(\zeta) = f(z) + \varphi(\zeta, z)$$

where $|\varphi(\zeta, z)| < \epsilon$, $|\zeta - z|$ being sufficiently small. Therefore,

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\cdot\int^{z+h}\varphi(\zeta, z)d\zeta\right| < \frac{1}{|h|}\epsilon\cdot|h| = \epsilon,$$

i.e.

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = F'(z) = f(z).$$

Hence, $F(z)$ is an analytic function of z with $f(z)$ as its derivative. By Art. 12 this derivative is again an analytic function, which had to be proven.

15. Representation of an Analytic Function by Means of Cauchy's Integral Formula.

If $f(\zeta)$ is continuous at all points ζ of a curve (C), then, *for any point z not located on (C),*

$$\varphi(z)=\frac{1}{2\pi i}\int\limits_{(C)}\frac{f(\zeta)d\zeta}{\zeta-z} \tag{64}$$

is an analytic function. For, if also $z+h$ is a point not located on (C), and if we form

$$\frac{\varphi(z+h)-\varphi(z)}{h}=\frac{1}{2\pi i}\int\limits_{(C)}\frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-z-h)},$$

it may be shown (as we did in Art. 12) that

$$\lim_{h\to 0}\frac{\varphi(z+h)-\varphi(z)}{h}=\frac{1}{2\pi i}\int\limits_{(C)}\frac{f(\zeta)d\zeta}{(\zeta-z)^2}.$$

But this integral exists because the function to be integrated is continuous along the whole path of integration. Therefore, $\varphi(z)$ possesses a derivative for any point not located on (C) and thus is analytic outside of (C).

As a matter of fact, it is not true to say that for closed curves (C) the "boundary values" $\varphi(\zeta)$, if at all existent, are identical with the values $f(\zeta)$. This is not always the case. For example, if $f(z)$ is analytic within a region (B) having (C) as its boundary line (z being located outside of (C)), then $\dfrac{f(\zeta)}{\zeta-z}$ is likewise analytic for all points within (B) and on (C); hence, by the principal theorem of Cauchy, equation (35),

$$\varphi(z)=\frac{1}{2\pi i}\oint\limits_{(C)}\frac{f(\zeta)d\zeta}{\zeta-z}=0$$

showing that the boundary values of $\varphi(z)$ most certainly are not identical with the values of $f(\zeta)$.

D. POWER SERIES IN THE COMPLEX PLANE

1. Infinite Series of Complex Terms in General.

Let

$$w_1, w_2, w_3 \ldots$$

represent an infinite succession of complex quantities $w_\lambda = u_\lambda + iv_\lambda$ $(\lambda = 1, 2, 3, \ldots)$. If the infinite series $u_1 + u_2 + \cdots$ as well as the infinite series $v_1 + v_2 + \cdots$, both having real terms, are convergent and if their sums are u and v, the series with complex terms $w_1 + w_2 + \cdots$ is said to be convergent and its sum to be $u + iv$, i.e. we have

$$\lim_{n \to \infty} (w_1 + w_2 + \cdots w_n) = u + iv.$$

The following auxiliary theorem holds for series with complex terms: if $|w_1| + |w_2| + |w_3| + \cdots$ is convergent, then $w_1 + w_2 + w_3 + \cdots$ is also convergent. For, $|w_\lambda| = \sqrt{u_\lambda^2 + v_\lambda^2} \geqq |u_\lambda|$, and also $|w_\lambda| \geqq |v_\lambda|$. The series $|w_1| + |w_2| + \cdots$ is thus a dominant series of the series $|u_1| + |u_2| + \cdots$ as well as of the series $|v_1| + |v_2| + \cdots$ Hence, the series is convergent and, therefore, $u_1 + u_2 + \cdots$ and $v_1 + v_2 + \cdots$ are also convergent and so is the series $w_1 + w_2 + \cdots$

2. Uniform Convergence.

Let the individual terms w_λ be functions of the complex variable z. If the series $w_1 + w_2 + \cdots$ is convergent, its sum $\varphi(z)$ is also a function of z:

$$w_1(z) + w_2(z) + w_3(z) + \cdots = \lim_{n \to \infty} \left(w_1(z) + w_2(z) + \cdots w_n(z)\right) = \varphi(z)$$

This means, more exactly, the following: Let

$$r_n(z) = \varphi(z) - \left(w_1(z) + w_2(z) + \cdots w_n(z)\right) = w_{n+1}(z) + w_{n+2}(z) + \cdots$$

represent the remainder after the first n terms of the series; then, after choosing a real arbitrarily small positive number ϵ, it is possible to determine a likewise real positive quantity $N = N(\epsilon, z)$ such that

$$|r_n(z)| < \epsilon \quad \text{for all } n > N(\epsilon, z).$$

The upper limit N, determining the minimum number of terms which have to be added up in order to force the remainder r_n of the series below a given value ϵ, generally depends on the said value ϵ *as well as on the argument* z. But if for all values of z of a region of the z-plane N does not depend on z but only on ϵ, this series is said to *converge uniformly* within this region. The conception of uniform convergence is particularly important for the following considerations. Some properties of uniformly convergent series are already well known from the elementary theory of series with real terms; the proofs of those properties are liter-

ally transferable to complex series, and that is why we may omit to give the proofs, just stating the theorems.

(a) *Theorem* of *Weierstrass.* An infinite series

$$w_1(z)+w_2(z)+\cdots$$

is said to be uniformly convergent in a region (**B**) if

$$|w_\lambda(z)|\leq a_\lambda \qquad (\lambda=1, 2, 3, \cdots)$$

and the series with positive *fixed* real terms a_λ, $a_1+a_2+\cdots$, is convergent.

(b) A series $w_1(z)+w_2(z)+\cdots$, *uniformly* convergent in a region (**B**), the terms of the series being continuous functions of z in the region (**B**), has a sum in this region which likewise is a continuous function.

(c) A series $w_1(z)+w_2(z)+\cdots$, *uniformly* convergent in a given region, the terms of the series being continuous functions of z in the above region, may be integrated term by term along any curve (C) located inside of the said region. This means the following: If we have

$$f(z)=w_1(z)+w_2(z)+\cdots,$$

we also have

$$\int_{(C)} f(z)dz=\int_{(C)} w_1(z)dz+\int_{(C)} w_2(z)dz+\cdots.$$

Within the region of uniform convergence we consequently may interchange summation of the series with integration of it, the latter being permitted term by term.

3. Representation of an Analytic Function by Means of a Uniformly Convergent Series of Given Analytic Functions (Weierstrass' Theorem of Double Series).

Let $f_\lambda(z)\,(\lambda=1, 2, 3, \cdots)$ be an infinite number of functions, analytic and one-valued throughout a region (**B**) and on its boundary (C).

If the series

$$f(z)=f_1(z)+f_2(z)+f_3(z)+\cdots \tag{65}$$

is uniformly convergent throughout (**B**) and on (C), its sum, $f(z)$, likewise represents a one-valued and analytic function in this region. The derivative of $f(z)$ may be determined in any component region ($\mathbf{B}_1$), completely located inside of (**B**), by means of the series

$$f'(z)=f'_1(z)+f'_2(z)+f'_3(z)+\cdots, \tag{66}$$

this series likewise being uniformly convergent in the region ($\mathbf{B}_1$). Under the assumptions made we, therefore, may differentiate series (65) term by term.

Proof: The series $\Sigma f_\lambda(z)$, being uniformly convergent in (**B**) and on (C) and consisting of continuous terms, its sum, by Art. 2b, is likewise

continuous and the series itself, by Art. 2c, may be integrated term by term. Consequently, by Cauchy's theorem

$$\oint_{(C)} f(\zeta)d\zeta = \oint_{(C)} \sum_{\lambda} f_{\lambda}(\zeta)d\zeta = \sum_{\lambda} \oint f_{\lambda}(\zeta)d\zeta = 0,$$

as all functions $f_{\lambda}(z)$ were assumed to be analytic on (C). The same result is obtained if the integration is extended along any simply closed curve completely within (**B**) instead of along the boundary (C). By the theorem of Morera (Sect. C, Art. 14), $f(z)$ thus represents an analytic function in (**B**). To determine its derivative, note that also the series

$$\frac{f(\zeta)}{(\zeta - z)^2} = \sum_{\lambda} \frac{f_{\lambda}(\zeta)}{(\zeta - z)^2} \tag{67}$$

converges uniformly along (C), but that the point z must not lie on (C) but inside of (**B**). This series, therefore, may also be integrated term by term along (C) which, by equation (55), proves the asserted formula (66). But the remainder after the first n terms of this series is

$$\rho_n(z) = f'_{n+1}(z) + f'_{n+2}(z) + \cdots = \frac{1}{2\pi i} \oint_{(C)} \frac{d\zeta}{(\zeta - z)^2} \left(f_{n+1}(\zeta) + f_{n+2}(\zeta) + \cdots\right)$$

$$= \frac{1}{2\pi i} \oint_{(C)} \frac{d\zeta}{(\zeta - z)^2} r_n(\zeta),$$

where $r_n(\zeta)$ represents the remainder after the first n terms of the given series $\Sigma f_{\lambda}(\zeta)$ on (C). Due to the uniform convergence of this series, however, $|r_n(\zeta)| < \epsilon$. If we denote by δ the smallest of all values assumed by $|\zeta - z|$, z remaining inside of (**B**) while ζ follows the boundary (C) (Fig. 13), we have by equation (40)

$$|\rho_n(z)| < \frac{1}{2\pi} \frac{\epsilon}{\delta^2} \cdot L = \epsilon_1, \tag{68}$$

where L is the length of arc of curve (C). As ϵ was arbitrarily small, ϵ_1 too has to be chosen arbitrarily small. Now, if we restrict z to an arbitrary region ($\mathbf{B}_1$), the boundary line (C_1) of which takes its course completely inside of (C), the same conclusions hold if only δ is understood to be the smallest distance between the points of ($\mathbf{B}_1$) and those on (C). The above demonstrates the uniform convergence of the series $\Sigma f'_{\lambda}(z)$ in any closed region ($\mathbf{B}_1$) completely located inside of (**B**). As a matter of fact, the same conclusions hold also for the derivatives of any order; all we have to do is to substitute a higher power for $(\zeta - z)^2$ of equation (67).

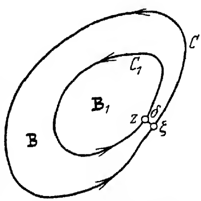


Fig. 13. The smallest distance between two boundary lines.

We shall later prove that any analytic function may be represented by a power series. In this sense formula (65) represents a series of power series, this circumstance explaining the designation "theorem of double series."

4. Generalization.

We shall now state and prove a far-reaching generalization of the theorem of double series and see that the *uniform convergence of the series* $\Sigma f_\lambda(z)$ *on the boundary* (C) *of the region* $(\mathbf{B})$ *is already sufficient* to deduce all other statements of the theorem. The proof should now, of course, be conducted in an entirely different manner, at the same time pointing out in a most beautiful way the range and the power of the methods of theory of functions. Let us assume again that $f_\lambda(z)$ $(\lambda = 1, 2, \ldots)$ is analytic and one-valued throughout $(\mathbf{B})$ and on the boundary (C); furthermore that the series $f_1(\zeta) + f_2(\zeta) + \ldots$ converges uniformly along the boundary (C). The assertions are: 1. the series $\Sigma f_\lambda(z)$ converges in any region $(\mathbf{B}_1)$ with the boundary (C_1), said region being located completely within (C), 2. the sum of the series is analytic in $(\mathbf{B}_1)$, 3. the series is uniformly convergent in $(\mathbf{B}_1)$, 4. the derivative of the series in $(\mathbf{B}_1)$ is obtained by differentiation of the individual terms: $\Sigma f'_\lambda(z)$, 5. the series composed of above derivatives is likewise uniformly convergent in $(\mathbf{B}_1)$, 6. the series possesses derivatives of arbitrary order in $(\mathbf{B}_1)$.

Proof. Let ζ be a variable on the boundary (C) of $(\mathbf{B})$. As the $f_\lambda(\zeta)$-values are analytic within this region and, therefore, continuous and as $\Sigma f_\lambda(\zeta)$ converges uniformly, the sum $f(\zeta) = \Sigma f_\lambda(\zeta)$ is also a continuous function of ζ. Now, let z be a point inside of $(\mathbf{B})$. Then, the series

$$\frac{f(\zeta)}{\zeta - z} = \sum_\lambda \frac{f_\lambda(\zeta)}{\zeta - z}$$

is also uniformly convergent along the boundary (C) (z being a fixed value), and consists of continuous terms only and thus has a continuous sum. We, therefore, may apply term by term integration by Art. 2, c and get:

$$\frac{1}{2\pi i}\oint_{(C)} \frac{f(\zeta)d\zeta}{\zeta - z} = \sum_\lambda \frac{1}{2\pi i}\oint_{(C)} \frac{f_\lambda(\zeta)d\zeta}{\zeta - z}.$$

But as the $f_\lambda(z)$-values are analytic throughout the inside of (C), Cauchy's integral formula (47) may be applied to the right-hand part of above equation, $f_\lambda(z)$ being the λ-th term. Hence,

$$\frac{1}{2\pi i}\oint \frac{f(\zeta)d\zeta}{\zeta - z} = \sum f_\lambda(z).$$

By Sect. C, Art. 15, the left-hand side represents a definite function $\varphi(z)$, and thus assertion 1 is proven. The function $\varphi(z)$ also is analytic in a region in which any point z is different from the points of the path of integration (C). This occurs for the region $(\mathbf{B}_1)$ including its boundary (C_1), indicating, that assertion 2 is also correct. The remainder after the first n terms of the series (see last equation) being designated by $\rho_n(z)$, the one of the series $\Sigma f_\lambda(\zeta)$ (uniformly convergent by assumption), being designated by $r_n(\zeta)$, we can state: $|r_n(\zeta)| < \epsilon$ for all sufficiently large values of n and for all points ζ on (C). All we have to do is to write $\rho_n(z)$ in the form

$$\rho_n(z) = f_{n+1}(z) + f_{n+2}(z) + \dots = \frac{1}{2\pi i} \oint\limits_{(C)} \frac{d\zeta}{\zeta - z} (f_{n+1}(\zeta) + f_{n+2}(\zeta) + \dots)$$

$$= \frac{1}{2\pi i} \oint\limits_{(C)} \frac{d\zeta}{\zeta - z} r_n(\zeta)$$

in order to realize, in the same manner as we did in equation (68), that $|\rho_n(z)|$ may be made

$$|\rho_n(z)| < \frac{1}{2\pi} \cdot \frac{\epsilon}{\delta} \cdot L = \epsilon_1,$$

which proves the uniform convergence of $\Sigma f_\lambda(z)$ in $(\mathbf{B}_1)$ and on (C_1) and, therefore, assertion 3. We now have

$$\varphi(z) = \Sigma f_\lambda(z),$$

i.e. a uniformly convergent series in $(\mathbf{B}_1)$ and on its boundary (C_1), the series being composed of analytic functions. Thus, the theorem of double series may be applied, from which the remaining assertions follow without any difficulties.

5. Power Series.

A series of the form

$$a_0 + a_1(z-a) + a_2(z-a)^2 + \dots,$$

where a, a_0, a_1, a_2, ... are fixed complex quantities, is called a power series. By substituting z in place of $z-a$, i.e. by shifting the starting point to the point $z=a$, the above power series may always be transformed into the form

$$\mathrm{P}(z) = \sum_{\lambda=0}^{\infty} a_\lambda z^\lambda, \tag{69}$$

A few simple theorems may be proved in the complex plane exactly in the same manner as in the real plane:

(a) If a power series converges for an arbitrary value $z=z_0$, it converges absolutely for all values $|z|<z_0$, i.e. at all points of the complex plane located inside of a *circle*, the center of which is the origin and which goes through point z_0.

(b) Any power series $\mathbf{P}(z)$ either converges only for $z=0$, or for all complex values of z (continuously convergent power series), or there exists a positive number r (the *radius of convergence*) of such properties that $\mathbf{P}(z)$ is convergent for all $|z|<r$ and divergent for all $|z|>r$. Note also the relation:

$$r=\frac{}{\limsup\limits_{n\to\infty}\sqrt[n]{a_n}}.$$

Examples.

$\mathbf{P}(z)=1+1!\,z+2!\,z^2+3!\,z^3+\ldots$ converges only for $z=0$;

$\mathbf{P}(z)=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots$ converges for all z;

$\mathbf{P}(z)=1+z+z^2+z^3+\ldots$ converges only for $|z|<1$.

The second series is the continuously convergent power series with the sum e^z, the third one — the geometric series with the sum $\frac{1}{1-z}$ and the radius of convergence $r=1$.

(c) The circle with the radius of convergence as its radius is called the *circle of convergence*. On this circle at least one point exists at which the power series (69) is no longer convergent.

(d) If two power series $\Sigma a_\lambda z^\lambda$ and $\Sigma b_\lambda z^\lambda$ have the same value for all values $|z|<r$, they are identical, i.e. $a_0=b_0$, $a_1=b_1$, $a_2=b_2$, ...

(e) *Uniform convergence of a power series.* Any power series converges uniformly inside of and on the boundary of any circle about the origin, the radius of which is *smaller* than the radius of convergence; the circle, therefore, must be located *inside* of the circle of convergence and must be concentric to the latter.

Proof: The above theorem follows directly from Weierstrass' theorem (Art. 2, a). For, if r represents the radius of convergence and ρ the radius of the inner concentric circle, and if

$$\rho<|z_0|<r,$$

then the series $\mathbf{P}(z_0)=\Sigma a_\lambda z_0^\lambda$ converges absolutely and $\Sigma\,|a_\lambda|\,|z_0^\lambda|$ is a convergent dominant series with fixed values for the series $\mathbf{P}(z)$, where $|z|\leq\rho$.

6. Expansion of an Analytic Function in a Power Series (Taylor's Series).

For real values of z and h the well known Taylor's expansion

$$f(z+h)=f(z)+\frac{f'(z)}{1!}h+\frac{f''(z)}{2!}h^2+\cdots$$

holds if $f(z)$ may be differentiated arbitrarily many times for real z values and if $\lim\limits_{n\to\infty} R_n(z, h)=0$, where $R_n(z, h)$ is the n^{th} remaining term of Taylor's formula. In order to transfer this series into the complex plane, substitute in Cauchy's integral, formula (47), $z=z_0+h$:

$$f(z_0+h)=\frac{1}{2\pi i}\oint\limits_{(C)}\frac{f(\zeta)d\zeta}{\zeta-z_0-h}.$$

It should be assumed that $f(z)$ is wholly analytic in a circle (K) with the boundary (C) and that the points z_0 and z_0+h are contained inside of this circle. Now, let us expand $\dfrac{1}{\zeta-z_0-h}$ into a series:

$$\frac{1}{\zeta-z_0-h}=\frac{1}{(\zeta-z_0)\left(1-\dfrac{h}{\zeta-z_0}\right)}=\frac{1}{\zeta-z_0}\left[1+\frac{h}{\zeta-z_0}+\frac{h^2}{(\zeta-z_0)^2}+\cdots\right.$$

$$\left.+\frac{h^n}{(\zeta-z_0)^n}+\frac{h^{n+1}}{(\zeta-z_0)^n(\zeta-z_0-h)}\right]. \tag{70}$$

Hence,

$$\frac{1}{2\pi i}\oint\frac{f(\zeta)d\zeta}{\zeta-z_0-h}=\frac{1}{2\pi i}\left[\oint\frac{f(\zeta)d\zeta}{\zeta-z_0}+h\oint\frac{f(\zeta)d\zeta}{(\zeta-z_0)^2}\right.$$

$$+h^2\oint\frac{f(\zeta)d\zeta}{(\zeta-z_0)^3}+\cdots+h^n\oint\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

$$\left.+h^{n+1}\oint\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}(\zeta-z_0-h)}\right]. \tag{71}$$

All integrals should be extended over the boundary (C).

By Cauchy's estimation formula we may show, however, that

$$\lim_{n\to\infty} h^{n+1}\oint\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}(\zeta-z_0-h)}=0.$$

On the other hand the remaining integrals are (by equations (55), (56),

(57)), proportional to the derivatives of $f(z)$ with the correspondent factorials, thus proving Taylor's series:

$$f(z_0+h)=f(z_0)+hf'(z_0)+\frac{h^2}{2!}f''(z_0)+\ldots \tag{72}$$

or

$$f(z)=f(z_0)+(z-z_0)f'(z_0)+\frac{(z-z_0)^2}{2!}f''(z_0)+\ldots \tag{73}$$

This series converges inside of a circle with z_0 as its center, the radius of the circle being equal to the distance from z_0 to the nearest point, at which $f(z)$ is no longer analytic. By equation (57) we may also represent Taylor's series as follows:

$$f(z_0+h)=\sum_{\lambda=0}^{\infty} a_\lambda h^\lambda, \tag{74}$$

where

$$a_\lambda=\frac{1}{2\pi i}\oint_{(C)}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{\lambda+1}} \qquad (\lambda=0, 1, 2, 3, \ldots) \tag{75}$$

Thus, the coefficients may also be determined by means of integration.

The fact that any analytic function of a complex variable, i.e. a function which may be differentiated, can be expanded in a power series has no parallel in the function theory of reals. In the real plane, even the existence of all derivatives is not sufficient for this purpose, as the example $f(z)=\exp\left(\frac{-1}{x^2}\right)$ shows.

7. Analytic Continuation.

Let $f(z)$ be expanded in a power series in powers of $(z-z_0)$,

$$f(z)=\sum_{\lambda=0}^{\infty} a_\lambda(z-z_0)^\lambda, \tag{76}$$

where the quantities a_λ are determined by formula (75) and where z is an inside point of the circle (K_0) with the center z_0, the circle being bounded by curve (C) and $f(z)$ being regular throughout this circle. Further let z be another inside point of the above circle; then, $f(z)$ evidently may also be expanded in powers of $(z-z_1)$ and this expansion holds for a circle (K_1) with the center z_1, if only $f(z)$ is regular throughout this circle. Now, it is quite possible that this circle (K_1) extends over the previous circle (K_0) (Fig. 14). The analytic function $f(z)$ is said to be continued analytically over the original region of expansion. Proceeding in this manner, we may represent the function $f(z)$ in a region

of the z-plane, under circumstances even in the total plane, by means of power series, each one of which (by Weierstrass) is called a functional element.

Let us explain the situation by means of a simple example, $f(z)=\frac{1}{z}$.

If (K_0) does not surround the origin we get the following expansion in powers of $z-z_0$:

$$a_\lambda=\frac{1}{2\pi i}\oint\frac{d\zeta}{\zeta(\zeta-z_0)^{\lambda+1}}=(-1)^\lambda\cdot\frac{1}{z_0^{\lambda+1}},$$

and hence

$$\frac{1}{z}=\frac{1}{z_0}-\frac{z-z_0}{z_0^2}+\frac{(z-z_0)^2}{z_0^3}-+\cdots \tag{77}$$

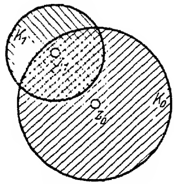


Fig. 14. Analytic continuation.

This series is convergent inside of a circle (K_0) with the center z_0, the origin being located on its boundary, i.e. the series is convergent in the region

$$|z-z_0|<|z_0|$$

of the circle (K_0). For $z=0, f(z)$ is no longer analytic. If we now choose a point z_1 inside of this circle and different from z_0, we can expand quite analogously

$$\frac{1}{z}=\frac{1}{z_1}-\frac{z-z_1}{z_1^2}+\frac{(z-z_1)^2}{z_1^3}-+\cdots$$

This series is convergent in the region

$$|z-z_1|<|z_1|$$

of the circle (K_1), which region, with appropriate choice of z_1, only partly overlaps the region (K_0). We even may choose a point z_2 such that the region of convergence (K_2), $|z-z_2|<|z_2|$, does not have a single point in common with (K_0), viz. if the circle (K_2) touches the circle (K_0) at the origin. The possibility to cover the complete plane by means of such overlapping circular regions is thus realized. The function $\frac{1}{z}$ is represented by a convergent power series in each of those circles.

Another example is the function $f(z)=z^m$ (integer m). The coefficients of Taylor's expansion of z^m in powers of $z-z_0$ have been determined in equation (60). We therefore have

$$z^m=z_0^m+\binom{m}{1}z_0^{m-1}(z-z_0)+\binom{m}{2}z_0^{m-2}(z-z_0)^2+\ldots \tag{78}$$

and this series stops at the term $(z-z_0)^m$ and represents an integer rational function for $m \geqq 0$; but if $m<0$, the series converges and holds for $|z-z_0|<|z_0|$. Consequently, for $m<0$, the origin must not be contained in the circle of convergence, having z_0 as its center; the circle of convergence, rather, goes through the origin, as $z=0$ is the only point at which for $m<0$ $f(z)$ stops to be analytic. We may perform the analytic continuation of the function $f(z)=z^m$ in the same manner as we previously did for $f(z)=\dfrac{1}{z}$.

E. LAURENT'S SERIES. RESIDUE THEOREMS. SINGULAR POINTS

1. Laurent's Series.

Let $f(z)$ be an analytic function in the annular region (B) including its boundary, (B) being formed by two concentric circles (K) and (K') about the center z_0 (Fig. 15). Let further $z=z_0+h$ be located inside of (B). Cauchy's integral theorem, applied to this annular region, yields

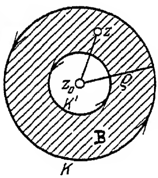


Fig. 15. Annular region

$$\frac{1}{2\pi i}\oint \frac{f(\zeta)d\zeta}{\zeta-z_0-h}=f(z_0+h),$$

the integration being extended over the entire boundary of (B) composed of (K) and (K').
Hence,

$$f(z_0+h)=\frac{1}{2\pi i}\left[\oint_{(K)}-\oint_{(K')}\right], \tag{79}$$

where both circles have to be taken in a positive direction. Each integral is calculated individually. Now, as in equation (70), we have first

$$\frac{1}{\zeta-z_0-h}=\frac{1}{\zeta-z_0}+\frac{h}{(\zeta-z_0)^2}+\frac{h^2}{(\zeta-z_0)^3}+\cdots$$
$$+\frac{h^n}{(\zeta-z_0)^{n+1}}+\frac{h^{n+1}}{(\zeta-z_0)^{n+1}(\zeta-z_0-h)}$$

and then, interchanging $\zeta-z_0$ and h,

$$-\frac{1}{\zeta-z_0-h}=\frac{1}{h}+\frac{\zeta-z_0}{h^2}+\frac{(\zeta-z_0)^2}{h^3}+\cdots+\frac{(\zeta-z_0)^n}{h^{n+1}}-\frac{(\zeta-z_0)^{n+1}}{h^{n+1}(\zeta-z_0-h)}.$$

Substituting this expansion for $f(z_0+h)$ of equation (79), we obtain

$$f(z_0+h)=\sum_{\lambda=-(n+1)}^{n} a_\lambda h^\lambda+R_n'+R_n'',$$

where the a_λ's are definite coefficients to be explained later, while R_n' and R_n'' represent the following integrals (remainder integrals):

$$R_n' = \frac{1}{2\pi i}\oint_{(K)} h^{n+1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}(\zeta-z_0-h)},$$

$$R_n'' = \frac{1}{2\pi i}\oint_{(K')} \frac{(\zeta-z_0)^{n+1}f(\zeta)d\zeta}{h^{n+1}(\zeta-z_0-h)}.$$

As the integration of R_n' refers to the circle (K), $\zeta-z_0$ as well as $\zeta-z_0-h$ can not vanish along the path of integration, because the points z_0 and z_0+h lie inside of (K). As in addition $f(\zeta)$ is analytic on (K), $|f(\zeta)|$ is restricted and therefore,

$$\frac{f(\zeta)}{|\zeta-z_0-h|}<M,$$

where M represents a real positive constant. Designating the radius of (K) with ρ, we have:

$$\frac{|h|}{|\zeta-z_0|}=\frac{|h|}{\rho}<1.$$

By Cauchy's estimation formula, equation (40), we thus have for R_n':

$$|R_n'| \leqq \frac{1}{2\pi}\cdot\frac{|h|^{n+1}}{\rho^{n+1}}\cdot M\cdot 2\pi\rho = \frac{|h|^{n+1}}{\rho^{n+1}}\cdot M\rho.$$

For $n\longrightarrow\infty$, however, $\frac{|h|^{n+1}}{\rho^{n+1}}\longrightarrow 0$. Hence,

$$\lim_{n\to\infty} R_n' = 0. \tag{80}$$

Analogously, we may show that

$$\lim R_n'' = 0 \tag{81}$$

and finally obtain:

$$f(z_0+h) = \sum_{\lambda=-\infty}^{+\infty} a_\lambda h^\lambda = \sum_{\lambda=-\infty}^{+\infty} a_\lambda (z-z_0)^\lambda. \tag{82}$$

In this equation the coefficients a_λ have the following values:

$$a_\lambda = \frac{1}{2\pi i}\cdot\oint_{(K)}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{\lambda+1}} \quad (\lambda = 0, +1, +2, +3, \ldots). \tag{83}$$

and

$$a_\lambda = \frac{1}{2\pi i}\cdot\oint_{(K')}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{\lambda+1}} \quad (\lambda = -1, -2, -3, -4, \ldots). \tag{84}$$

By reason of Cauchy's integral theorem, it is evident that any simply closed curves (C) and (C') may be taken in place of the circles (K) and (K') in equations (83) and (84) respectively; these curves must, however, enclose the point z_0, and the function $f(z)$ must be regular within the annular regions between (C) and (K) and between (C') and (K'). Inside of (C) and outside of (C') and, in particular, at the point z_0 it is by no means required that $f(z)$ be regular.

Formula (82), in connection with formulas (83) and (84), is called *Laurent's series.* If all a_λ's as determined by equation (84) have the value zero, equation (82) goes over in *Taylor's series,* equation (74).

2. Examples of Laurent's Series Expansions.

(a) The rational fractional function

$$\frac{1}{z(z-1)}$$

is regular at all points with exception of the points $z=0$ and $z=1$. Therefore, this function may be expanded either in an annular region about $z=0$ or about $z=1$. In the first case, using the geometric series, equation (77), we get

$$\frac{1}{z(z-1)} \quad \frac{1}{z} \quad \frac{1}{1-z} = -\frac{1}{z}\cdot(1+z+z^2+\cdots),$$

i.e.

$$\frac{1}{z(z-1)} = -\frac{1}{z} - 1 - z - z^2 - \cdots \qquad (z\neq 0) \tag{85}$$

In the second case we may write

$$\frac{1}{z(z-1)} \quad \frac{1}{z-1} \quad \frac{1}{(z-1)+1} \quad \frac{1}{z-1}(1-(z-1)+(z-1)^2-+\cdots),$$

i.e.

$$\frac{1}{z(z-1)} = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + - \cdots \qquad (z\neq 1) \tag{86}$$

In both expansions only one term with negative index $(\lambda = -1)$ appears.

(b) The function

$$e^{\frac{1}{z}}$$

may be expanded in a series in any annular region enclosing the origin, the series being

$$e^{\frac{1}{z}} = 1 + \frac{1}{z\cdot 1!} + \frac{1}{z^2\cdot 2!} + \frac{1}{z^3\cdot 3!} + \cdots \qquad (z\neq 0) \tag{87}$$

Here, *all* the terms of Laurent's series, having a non-positive index, and *only* such terms, are present.

3. Residues.

For the coefficient a_{-1} of Laurent's series expansion the following formula holds:

$$a_{-1}=\frac{1}{2\pi i}\oint_{(K')} f(\zeta)d\zeta. \tag{88}$$

This complex quantity is called the *residue* of $f(z)$ at the point $z=z_0$ and is denoted by the following symbol:

$$a_{-1}=\text{Res}\,f(z)_{z=z_0}. \tag{89}$$

If the Laurent's series for $f(z)$ is known in the vicinity of $z=z_0$ or is easier to compute than the integral, we may make use of the residue in order to determine the integral. A very elementary example is $f(z)=\dfrac{1}{z}$: here, the integral taken around the origin is

$$\oint \frac{dz}{z}=2\pi i\cdot 1,$$

the Laurent's series in this case consisting of the only term $\dfrac{1}{z}$ with the residue $a_{-1}=1$. Compare also the derivation of formula (38).

Just as easily it may be proved that

$$\oint e^{\frac{x}{z}}\,dz=2\pi ix, \tag{90}$$

if the path of integration encloses the origin but once. For, by equation (87), we have

$$e^{\frac{x}{z}}=1+\frac{x}{z}+\frac{x^2}{z^2 2!}+\frac{x^3}{z^3 3!}+\cdots,$$

and therefore

$$\text{Res}\left(e^{\frac{x}{z}}\right)_{z=0}=x,$$

from which equation (90) follows at once.

As we see, the formula

$$\frac{1}{2\pi i}\oint f(\zeta)d\zeta=\text{Res}\,f(z)_{z=z_0} \tag{91}$$

is an extension of Cauchy's integral theorem, equation (35).

4. The Residue Theorem.

Let $f(z)$ be an analytic function inside of a region (B) at all points except at the points

$$z_1,\, z_2,\, \ldots\, z_n,$$

and let the function be also analytic at all points of the boundary (C) of $(\mathbf{B})$. Integrating along the path indicated in Fig. 16, on which path $f(z)$ is analytic at all points, we obtain

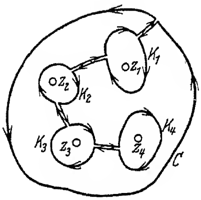


Fig. 16. Referring to the residue theorem.

$$\oint_{(C)} f(\zeta)d\zeta = \oint_{(K_1)} + \oint_{(K_2)} + \cdots + \oint_{(K_n)}.$$

Using the residue relation (91), we thus have

$$\frac{1}{2\pi i}\oint_{(C)} f(\zeta)d\zeta = \sum_{\lambda=1}^{n} \operatorname{Res} f(z)_{z=z_\lambda}. \tag{92}$$

The last formula is called the *residue theorem.* The summation on the right-hand side is to be extended over all points inside of (C), at which $f(z)$ ceases to be regular.

Example. For the function $f(z) = \frac{\quad}{z(z-1)}$ we have, for a path of integration enclosing the points 0 and 1, as indicated in Fig. 17:

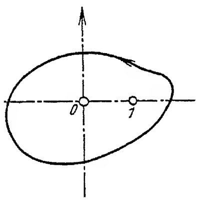


Fig. 17. Rotation about two singular points.

$$\frac{1}{2\pi i}\oint \frac{d\zeta}{\zeta(\zeta-1)} = -1+1=0.$$

This follows at once from the expansions (85) and (86), from which the residues of $f(z)$ at the points 0 and 1 should be obtained directly.

5. Singular Points of an Analytic Function.

All points of the z-plane at which an analytic function does not possess a derivative (is not regular) are said to be singular points of $f(z)$. Referring to *one-valued* functions, we distinguish the *poles* or *non-essential singular points* from the *essential singular points.* Let z_0 be a singular point. Expand $f(z)$ in a Laurent's series in powers of $z-z_0$; then, this expansion certainly contains powers of $z-z_0$ with negative exponents, for otherwise $z-z_0$ would not be a singular point. Two cases are possible.

(a) Let

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \tag{93}$$

where the powers of $z-z_0$ with negative exponents appear in finite number. Then, the point $z=z_0$ is said to be a *pole* of the function $f(z)$. If

$a_{-m} \neq 0$ and $-m$ is the absolutely largest of the negative exponents and if the function

$$f_1(z) = (z - z_0)^m f(z)$$

behaves regularly at the point z_0, m is called the *order of the pole* z_0, or, $f(z)$ is said to have a pole of m^{th} order at the point z_0.

The sum of all terms with negative exponents of $z - z_0$, i.e. the expression

$$\frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-m}}{(z - z_0)^m},$$

is called the "*principal part*" of the function.

(b) Let

$$f(z) = \sum_{\lambda=\infty}^{0} \frac{a_{-\lambda}}{(z - z_0)^\lambda} + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \tag{94}$$

where now the powers of $z - z_0$ with negative exponents appear in infinite number. Then, z_0 is said to be an *essential singular point*. It is now not possible to create a regular function at z_0 by multiplication of $f(z)$ by as high a power of $z - z_0$ as we may choose.

The rational function

$$f(z) = \frac{1}{z(z-1)^2},$$

for example, has a pole of the first order at $z = 0$ and a pole of the second order at $z = 1$. The function

$$f(z) = e^{\frac{1}{z}},$$

on the contrary, has, as may be realized by inspecting equation (87), an essential singular point at $z = 0$.

(c) A function not having (in the finite) any other singular points but poles, is said to be a *meromorphic* function, or a function of the nature of a rational function.

5a. A simple formula exists for the *relationship between the number of poles and zeros* which a one-valued analytic function $f(z)$ possesses inside of a region (**B**) in which this function otherwise is regular:

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = N - P. \tag{94a}$$

In this formula (C) is the boundary of the region, N and P are the number of zeros and poles respectively, which $f(z)$ possesses inside of

(**B**), each zero and each pole, however, being counted as many times as is indicated by the corresponding number of order. In the neighborhood of z_0 we have

$$f(z)=a_m(z-z_0)^m+a_{m+1}(z-z_0)^{m+1}+\cdots,$$

where $m>0$ if z_0 is a zero of m^{th} order and where, on the contrary, $m<0$ for a pole with the number of order $-m$. Furthermore,

$$f'(z)=a_m m(z-z_0)^{m-1}+a_{m+1}(m+1)(z-z_0)^m+\cdots$$

and, therefore,

$$\frac{f'(z)}{f(z)}=\frac{m}{z-z_0}+\mathbf{P}(z-z_0),$$

where $\mathbf{P}(z-z_0)$ indicates an ordinary power series.
Hence,

$$m=\operatorname{Res}\left(\frac{f'(z)}{f(z)}\right)_{z=z_0}$$

If we carry through this process for each of the n different zeros, the numbers of order of which may be designated by N_1, N_2, $\cdots$ N_n, and for each of the p different poles with the numbers of order P_1, P_2, $\cdots$ P_p, then the residue theorem, equation (92), applied to the function $\dfrac{f'(z)}{f(z)}$, yields directly the desired result, equation (94a), if we put

$$N_1+N_2+\cdots N_n=N,$$

$$P_1+P_2+\cdots P_p=P.$$

Example. The function $f(z)=\dfrac{(z+1)^2}{z(z-1)^2}$ has $N=2$ and $P=3$ in a region, inside of which the points $z=-1,\ 0,\ +1$ are located, and we have

$$\oint\frac{f'(z)}{f(z)}\,dz=\oint\frac{2dz}{z+1}-\int\frac{dz}{z}-\int\frac{2dz}{z-1}=-2\pi i.$$

Note. If $f(z)$ is regular throughout a region (**B**), i.e. if $P=0$, the integral (94a) gives the number of zeros in (**B**).

6. The "Infinitely Remote" Point.

The behavior of a one-valued analytic function $f(z)$ for absolute large values of z or, as we say, the behavior at the infinitely remote point, may be reduced to the behavior of another function $\varphi(\zeta)$ at the origin $\zeta=0$.

For, if we put

$$z = \frac{1}{\zeta},$$

we have $|\zeta| \to 0$ for $|z| \to \infty$.

Let

$$f(z) = f\left(\frac{1}{\zeta}\right) = \varphi(\zeta).$$

If $\varphi(\zeta)$ is regular at $\zeta = 0$, $f(z)$ is said to be regular in infinity. If $\varphi(\zeta)$ has a pole of m^{th} order or an essential singular point at $\zeta = 0$, the same conditions are assigned to the function $f(z)$ at the infinitely remote point.

Thus, it may be shown that $f(z) = \dfrac{1}{z}$ is regular in infinity; that $f(z) = z^2$ has a pole of second order "at $z = \infty$"; that $f(z) = e^z$, on the contrary, has an essential singular point in infinity; then, referring to the last function and comparing with equation (87),

$$f\left(\frac{1}{\zeta}\right) = \varphi(\zeta) = e^{\frac{1}{\zeta}} = 1 + \frac{1}{\zeta \cdot 1!} + \frac{1}{\zeta^2 \cdot 2!} + \cdots$$

has an essential singular point at $\zeta = 0$.

The designation "infinitely remote point of the z-plane" or "$z = \infty$" is, of course, to be understood only in the figurative sense and one should not attempt to visualize something intuitively geometrical. Talking about the behavior of an analytic function $f(z)$ in the "*completed*" complex plane, we mean its behavior for complex z-values and, moreover, for $|z| \to \infty$ in the above sense. In this connection, there exists a peculiar theorem stating that any analytic function, if at all susceptible of assuming different values, must possess a singular point at least at one point of the completed plane.

7. Liouville's Theorem.

A one-valued analytic function, regular throughout the completed complex plane, is a constant.

Proof. As $f(z)$ is supposed to be regular at all points, also at $z = \infty$, $|f(z)|$ is everywhere restricted, i.e. there exists a constant K such that

$$|f(z)| < K$$

for any z and also for $|z| \to \infty$. Now, let z and $z+h$ be two different points of the z-plane and let (C) be a circle about z as center, the radius R being

so large that $z+h$ still lies in the inside of the circle and within a distance of at least $\frac{1}{2}R$ from the boundary of the circle (Fig. 18). By Cauchy's integral formula (47) we have

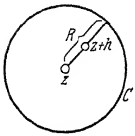


Fig. 18. Referring to Liouville's Theorem.

$$f(z+h)-f(z)=\frac{1}{2\pi i}\oint_{(C)}\left(\frac{1}{\zeta-z-h}-\frac{1}{\zeta-z}\right)f(\zeta)d\zeta$$

$$\frac{1}{2\pi i}\oint_{(C)}\frac{hf(\zeta)}{(\zeta-z-h)\,(\zeta-z)}d\zeta$$

and thus by equation (40):

$$|f(z+h)-f(z)|\leq\frac{1}{2\pi}\,\frac{|h|K}{\frac{1}{2}R\cdot R}\int^{2\pi}Rd\varphi=2\frac{|h|}{R}$$

But we may make R arbitrarily large and, therefore, the right-hand side arbitrarily small and this for any value of z and h. Thus,

$$f(z+h)=f(z).$$

Hence, $f(z)$ is constant, which was to be proven.

8. Remarks Concerning Many-Valued Functions.

We shall have to content ourselves with a few examples which, however, will point out clearly the peculiarities encountered with many-valued functions.

(a) $w=\sqrt{z-z_0}$. This function possesses (for any value of z different from z_0) two separate functional values which are distinguished as the two branches of the function, namely for $z-z_0=re^{i\varphi}$:

$$w_1=|\sqrt{r}|\,e^{\frac{i\varphi}{2}}$$

and

$$w_2=|\sqrt{r}|e^{i\left(\frac{\varphi}{2}+\pi\right)}.$$

Only for $z=z_0$, w has *one* value: zero. Therefore, $z=z_0$ is called a *branch-point* of w. To study the behavior of $w=f(z)=\sqrt{z-z_0}$ in the neighborhood of the branch point, introduce $w=\rho e^{i\vartheta}$ and let the point z circumscribe the point z_0 along a simply closed curve. Then $\varphi=\text{arc}\,(z)$ increases from an initial value φ_0 to $\varphi_0+2\pi$, while $\vartheta=\text{arc}\,(w)$ increases from the initial value $\vartheta_0=\dfrac{1}{2}\varphi_0$ to $\vartheta_0+\pi$. In this manner, starting from the value

of one branch, say w_1, we arrive at the values of the other branch w_2, and only after describing another closed path in the z-plane about the branch-point z_0 we regain the original value w_1. Thus, a double circuit in the z-plane about the point z_0 corresponds to a single circuit in the w-plane about the point $w=0$.

Riemann has introduced the conception of a multiple-sheeted (here, with $w=\sqrt{z-z_0}$, a double-sheeted) plane, using a separate z-plane for each of the two branches w_1 and w_2; these planes are put one atop the other in such a manner that the values of z coincide. Both sheets thus produced have the branch-point $z=z_0$ in common as $w_1=w_2$ at that point. In order to justify the fact that the initial values of w are obtained only after a double circuit about the branch-point $z=z_0$, imagine the two sheets to be fastened along any half-line starting from $z=z_0$ (or else any curve without multiple point going to infinity) in the manner indicated in Fig. 19. In this double-sheeted "*Riemann surface*" let us now start from one point of the straight line and describe a continuous path about the point z_0; we thus obtain the values of the one branch w_1 which are coördinated to the values of the upper sheet; arriving at the straight line again, we have to transfer to the lower sheet to which the values w_2 are coördinated; a repeated crossing of the straight line again transfers us to the upper sheet.

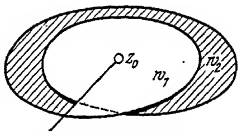


Fig. 19. Double-sheeted "Riemann surface."

(b) $w=\sqrt[m]{z-z_0}$. This function is, for $z\neq z_0$, m-valued and has a branch-point of m^{th} order at $z=z_0$, the corresponding Reimann surface consisting of m different sheets, the first being connected with the m^{th} along the "branch-cut" started from z_0.

(c) $w=\log z=\ln|z|+i\ \text{arc}\ (z)+2k\pi i\ (k=0,\pm1,\pm2,\ldots)$. This function is infinitely many-valued; $z=0$ is here a so-called *spiral-point* of infinitely high order, because the corresponding Riemann surface consists of an infinite number of sheets connected together at the point z_0.

It may be readily realized that the branch-points and the spiral-points pertain to the singular points; the function $w=\sqrt{z-z_0}$, for example, possesses

$$\frac{dw}{dz}=\frac{1}{\sqrt{z-z_0}}$$

only if $z\neq z_0$.

F. APPLICATIONS AND MISCELLANEOUS THEOREMS

1. Calculation of a Definite Integral.

Let it be required to calculate the integral

$$\oint e^{iz}\frac{dz}{z}$$

for the closed path indicated in Fig. 20 and, in particular, for the limiting case $R\to\infty$, $r\to 0$. The function e^{iz}/z being analytic inside the region in question, we have:

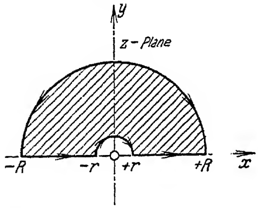


Fig. 20. Closed path of integration.

$$\oint e^{iz}\frac{dz}{z}=0. \tag{95}$$

We shall now split-up the integration into the following parts:

I. Along the semi-circle of radius R: $z=Re^{i\varphi}$; $dz=Re^{i\varphi}\cdot id\varphi=z\cdot id\varphi$.

II. Along the x-axis from $-R$ to $-r$: $z=x$; $dz=dx$.

III. Along the semi-circle of radius r: $z=re^{i\varphi}$; $dz=z\cdot id\varphi$.

IV. Along the x-axis from $+r$ to $+R$: $z=x$; $dz=dx$.

By equation (95), the sum of these four integrals must be equal to zero. We may estimate integral I as follows:

$$|I|=\left|\int_0^{\pi} e^{iR\cos\varphi}\cdot e^{-R\sin\varphi}d\varphi\right|\leq\left|\int_0^{\pi} e^{-R\sin\varphi}d\varphi\right|=2\int_0^{\pi/2} e^{-R\sin\psi}d\psi.$$

This integral may be further split-up into

$$\int_0^{\epsilon}+\int_{\epsilon}^{\pi/2},$$

where $\epsilon>0$ may be taken arbitrarily small; the first integral is at most equal ϵ, because of $e^{-R\sin\psi}\leq 1$ for $0\leq\psi\leq\epsilon$; the second integral is at most equal $e^{-R\sin\epsilon}\left(\frac{\pi}{2}-\epsilon\right)<\frac{\pi}{2}e^{-R\sin\epsilon}$, because of $e^{-R\sin\psi}\leq e^{-R\sin\epsilon}$. Therefore, combined:

$$|I|<2\left(\epsilon+\frac{\pi}{2}e^{-R\sin\epsilon}\right).$$

But R may be taken so large as to make $\frac{\pi}{2}e^{-R\sin\epsilon}<\epsilon$, the required relation being

$$R>\frac{\ln\frac{\pi}{2}-\ln\epsilon}{\sin\epsilon}. \tag{96}$$

In this case we have

$$|I| < 4\epsilon,$$

i.e. arbitrarily small. Hence

$$\lim_{R\to\infty} I = 0 \tag{97}$$

Applying the power series for e^z, integral III is found to be:

$$\begin{aligned} III &= -i\int_{\varphi=0}^{\pi} \exp(ire^{i\varphi})d\varphi \\ &= -i\int_{\varphi=0}^{\pi} \left\{ 1+ire^{i\varphi}+\frac{i^2r^2}{2!}e^{2i\varphi}+\frac{i^3r^3}{3!}e^{3i\varphi}+\dots \right\} d\varphi \end{aligned} \tag{98}$$

But the above series is uniformly convergent for all values of φ to be taken into consideration (compare Sect. D, Art. 2a) because the absolute value of the$(\lambda+1)^{\text{st}}$-term is $\leqq \frac{r^\lambda}{\lambda!}$ and the latter is independent of φ and equal to the $(\lambda+1)^{\text{st}}$ term of the convergent series for e^r with positive real terms. Consequently, the term by term integration of equation (98) is permissible. The involved integrals are of the form

$$\left.\begin{aligned} \int_0^{\pi} e^{\lambda i\varphi}d\varphi &= \frac{1}{\lambda i}(e^{\lambda i\pi}-1) \\ &= \frac{1}{\lambda i}((-1)^\lambda-1) \end{aligned}\right\} \quad (\lambda=0,1,2,\dots) \tag{99}$$

and thus equal 0 or $2i/\lambda$, depending upon λ being even or odd. We, therefore, have

$$III = -i(\pi+rS(r)), \tag{100}$$

where

$$S(r) = 2\left(-1+\frac{r^2}{3\cdot 3!}-\frac{r^4}{5\cdot 5!}+\frac{r^6}{7\cdot 7!}-+\dots\right) \tag{101}$$

represents a continuously convergent power series with $|S(r)| < 2|\cos r| \leqq 2$. From equation (100) we therefore conclude:

$$\lim_{r\to 0} III = -i\pi. \tag{102}$$

Finally, the integrals II and IV for $R\to\infty$ and $r\to 0$ may be comprised as follows:

$$\lim\,(II+IV) = \int_{-\infty}^{0} + \int_{0}^{-\infty} = \int_{\infty}^{+\infty} e^{ix}\frac{dx}{x}. \tag{103}$$

Adding up all integrals and considering equations (95), (97) and (102), the result for the limiting case $R \longrightarrow \infty$, $r \longrightarrow 0$ is:

$$\int_{-\infty}^{+\infty} e^{ix} \frac{dx}{x} = +i\pi. \tag{104}$$

Resolving equation (104) into the real and imaginary component, we have:

$$\int^{+\infty} \cos x \frac{dx}{x} = 0 \tag{105}$$

and

$$\int_{-\infty}^{+\infty} \sin x \frac{dx}{x} = \pi \tag{106}$$

or else

$$\int_0^{\infty} \sin x \frac{dx}{x} = \frac{\pi}{2}. \tag{107}$$

It is not quite simple to calculate the last integral without the aid of complex considerations.

2. Hook-Integrals.

The last equations, particularly equation (104), written in abbreviated notation, do not reproduce the situation in its full clearness, in as much as the origin, of course, has to be excluded. Keeping the path of integration as indicated in Fig. 20, considering, however, the limiting case $R \longrightarrow \infty$, we get from equations (95) and (97)

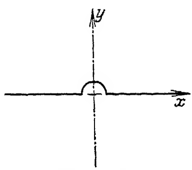


Fig. 21. Hook.

$$\int_{\frown\!\!\!\to} e^{iz} \frac{dz}{z} = 0, \tag{108}$$

where the hook indicates the path of integration (compare Fig. 21), the integral being called hook-integral.* As will be noted, we have gone around the origin following a circle of *arbitrary radius r*. Substituting

$$z = i\zeta, \qquad -iz = \zeta, \qquad \frac{dz}{z} = \frac{d\zeta}{\zeta},$$

* This notation originates from A. Korn.

the path of integration is rotated through $\operatorname{arc}(-i)=3\dfrac{\pi}{2}$ and the hook-integral of Fig. 22 is obtained:

$$\int_{\supset} e^{-\zeta}\frac{d\zeta}{\zeta}=0.$$

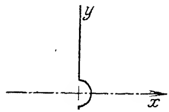


Fig. 22. Hook.

Reversing the direction of integration means a change in the sign of the integral; thus, also in Fig. 23,

$$\int_{\supset} e^{-\zeta}\frac{d\zeta}{\zeta}=0. \qquad (109)$$

Now let t be a *real* variable. If $t>0$, this only means a change in scale of the z-plane. Introducing $t\zeta$ in place of ζ in equation (109), we have:

$$\int_{\supset} e^{-t\zeta}\frac{d\zeta}{\zeta}=0 \text{ for } t>0. \qquad (110)$$

But if t is negative or zero the value of the integral changes. To find this value, let us first compute the integral

Fig. 23. Hook.

$$\frac{1}{2\pi i}\oint e^{iz}\frac{dz}{z}$$

for the path of integration as indicated in Fig. 24. As $z=0$ is the only singularity of $\dfrac{e^{iz}}{z}$ in the traversed region, the integral is, by equation (88), equal to

$$\operatorname{Res}\left(\frac{e^{iz}}{z}\right)_{z=0}.$$

In the neighborhood of $z=0$ the Laurent's series of $\dfrac{e^{iz}}{z}$ is $\dfrac{e^{iz}}{z}=\dfrac{1}{z}+\dfrac{i}{1!}+\dfrac{i^2}{2!}$ $+\dfrac{i^3}{3!}z^2+\cdot$ and, hence, the residue is equal to 1. Thus, for $R\longrightarrow\infty$ and considering equation (97), the hook-integral (Fig. 25) is:

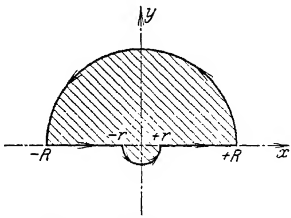


Fig. 24. Closed path of integration.

$$\frac{1}{2\pi i}\int e^{iz}\frac{dz}{z}=1. \qquad (111)$$

The substitution $iz=\zeta$ results in a rotation of the z-plane through $+\frac{\pi}{2}$ and we obtain

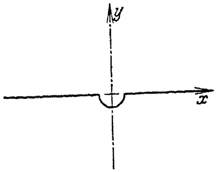


Fig. 25. Hook.

$$\int e^{\zeta}\frac{d\zeta}{\zeta}=2\pi i. \tag{112}$$

Substituting $t\zeta$ for ζ, t again being real and $t>0$, we get

$$\frac{e^{t\zeta}d\zeta}{\zeta}=2\pi i \text{ for } t>0. \tag{113}$$

The left side of this equation goes over into the left side of equation (110) if we substitute $-t$ for t, i.e. if we make t negative in equation (110). To complete the result for the case $t=0$, consider

$$\int\frac{d\zeta}{\zeta}.$$

For the path of Fig. 26 this integral vanishes, because there are no singularities of the function $\frac{1}{\zeta}$ in the enclosed region, thus:

$$\oint\frac{d\zeta}{\zeta}=0. \tag{114}$$

This closed integral may, however, be composed of the hook-integral extended from $-Ri$ to $+Ri$ (leaving the origin out of the path) and the arc-integral along the semi-circle (H) of radius R (Fig. 26). Thus,

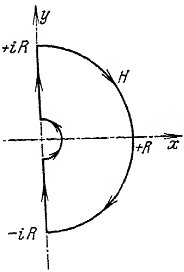


Fig. 26. Closed path of integration.

$$\int_{-Ri}^{+Ri}\frac{d\zeta}{\zeta}+\int_{(H)}\frac{d\zeta}{\zeta}=0.$$

On the semi-circle (H) we have $\zeta=R\cdot e^{i\varphi}$ and $d\zeta=i\zeta\cdot d\varphi$, and the second integral thus becomes:

$$\int_{(H)}\frac{d\zeta}{\zeta}=\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} i\,d\varphi=-i\pi\,.$$

This result being entirely independent of R, we may at once go over to the limit $R\to\infty$ and find:

$$\int\frac{d\zeta}{\zeta}=i\pi\,. \tag{115}$$

The integration has to be extended over the complete imaginary axis, deviating to the right in the neighborhood of the origin only.

3. Impulse Function.

The results of formulas (110), (113) and (115) may be written as follows:

$$\frac{1}{2\pi i}\int_{\mathfrak{I}} e^{t\zeta}\frac{d\zeta}{\zeta} = \begin{cases} 0 & \text{for } t<0 \\ \frac{1}{2} & \text{for } t=0 \\ 1 & \text{for } t>0. \end{cases} \tag{116}$$

The so determined real discontinuous function $S(t)$ is called a *discontinuous factor*. We shall call such a function, suddenly jumping from 0 to 1 when passing from negative to positive values of the argument (the fact that it assumes the intermediate value $\frac{1}{2}$ at the origin itself is not essential), an *impulse function*.* Such functions appear, for example, if we suddenly switch-in a current into an open circuited electric line; we speak of sending a current impulse into the line.

If we denote by sgn t (sign of t) the value $+1, 0, -1$, depending upon t being positive, zero or negative, $S(t)$ may easily be expressed by the function sgn t. It is

$$S(t) = \frac{1}{2\pi i}\int_{\mathfrak{I}} e^{t\zeta}\frac{d\zeta}{\zeta} = \frac{1}{2}(1+\operatorname{sgn} t). \tag{117}$$

Figure 27 shows the shape of this impulse function. An impulse function jumping from 0 for $t=t_0$ to a positive value K is simply

$$K\cdot S(t-t_0).$$

We shall return to this equation in one of the next articles.

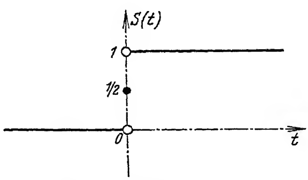


Fig. 27. Impulse function $S(t)$.

4. Linear Differential Equations and Heaviside Operational Calculus.

As is well known, the determination of elastic or harmonic oscillations depends upon the integration of one or several *linear* differential equations, which, depending upon the number of dimensions of the medium, may be ordinary or partial differential equations. The theory of periodic electric phenomena, such as the electro-magnetic oscillations in ether or the oscillating process of current

* Translator's remark: the English literature refers to this function as the "Unit-Function." The notation $S(t)$ for this function is derived from the German word "Stossfunktion."

and voltage in a cable or in a line, is also based on the study of such linear differential equations. We shall discuss here the simplest case of *ordinary linear* differential equations.

For many purposes this theory uses a symbolic notation for the necessary differentiations with respect to the independent variable. This notation, originated by Leibniz, is known since the eighteenth century (Lagrange) and has been discussed in detail in some of the older textbooks.* And yet, electrical engineers of today refer to it as the Heaviside notation.

Let t be the independent *real* variable (in physical applications it is mostly the time) and x a *real or complex* function of t, which may, however, be differentiated sufficiently many times. Let further

$$a_0, a_1, a_2, \cdots a_m$$

be given functions of t, or else, constants. Consider the linear differential expression of m^{th} order (if $a_m \neq 0$)

$$a_0 x + a_1 \frac{dx}{dt} + a_2 \frac{d^2x}{dt^2} + \cdots + a_m \frac{d^m x}{dt^m}. \tag{118}$$

Let us now introduce the following abbreviations

$$\frac{dx}{dt} = \frac{d}{dt} x = Dx, \quad \frac{d^2x}{dt^2} = \frac{d^2}{dt^2} x = D^2 x, \cdots \frac{d^m x}{dt^m} = D^m x \tag{119}$$

and let us consider Dx as a "product" of the "symbolic operator" D and x; actually, D in front of x is simply a "command" to differentiate x with respect to t. Similarly, consider D^m as the symbolic m^{th} power of D. Then, the linear differential expression (118) may be rewritten as follows:

$$L(D)x = (a_0 + a_1 D + a_2 D^2 + \cdots + a_m D^m)x, \tag{120}$$

where the "integer rational function of D"

$$L(D) = a_0 + a_1 D + a_2 D^2 + \cdots + a_m D^m \tag{121}$$

(a linear differential operator of m^{th} order) is nothing else but the "command" to treat the function x, which follows D, in the manner prescribed by equation (118). The equation

$$L(D)x = T(t) \tag{122}$$

thus is a homogeneous linear differential equation with the "disturbance function" $T(t)$. Suppose this function is proportional to the exciting acceleration; then, the integration of (122) produces the x-component of an oscillation which, in particular, is called harmonic if $a_0, a_1 \cdots a_m$ are constants. Let x_0 be a particular solution of equation (122), i.e.

$$L(D)x_0 = T(t);$$

*Z.B.G. Boole, "Treatise on the Calculus of Finite Differences," (Cambridge, 1860).

then, $x-x_0$ satisfies the homogeneous differential equation corresponding to equation (122), viz.

$$L(D)\ (x-x_0)=0 \tag{123}$$

The non-abbreviated form of equation (123) is:

$$a_0(x-x_0)+a_1\frac{d}{dt}(x-x_0)+a_2\frac{d^2}{dt^2}(x-x_0)+\cdots+a_m\frac{d^m}{dt^m}(x-x_0)=0.$$

5. Coupled Oscillations.

Consider a system of n such linear differential equations with n unknown functions $x_1, x_2 \cdots x_n$ of t and n known "exciting" functions $T_1, T_2, \cdots T_n$ of t; then we have:

$$\left.\begin{aligned} &L_{11}(D)x_1+L_{12}(D)x_2+\cdots+L_{1n}(D)x_n=T_1(t)\\ &L_{21}(D)x_1+L_{22}(D)x_2+\cdots+L_{2n}(D)x_n=T_2(t)\\ &\\ &L_{n1}(D)x_1+L_{n2}(D)x_2+\cdots+L_{nn}(D)x_n=T_n(t).\end{aligned}\right\} \tag{124}$$

This system may be written in the brief form

$$\sum_{k=1}^{n} L_{jk}(D)x_k=T_j(t), \qquad (j=1, 2, \cdots n) \tag{125}$$

where the $L_{jk}(D)$'s are n^2 *linear* differential operators of the form of equation (121), for instance

$$L_{jk}(D)=a_{jk,0}+a_{jk,1}D+a_{jk,2}D^2+\cdots+a_{jk,m_{jk}}D^{m_{jk}}, \qquad (j, k=1, 2, 3, \cdots n) \tag{126}$$

where m_{jk} designates the order of the operator $L_{jk}(D)$ and the coefficients $a_{jk,\lambda}$ are given functions of t or constants.

The n oscillating quantities, $x_1, x_2, \cdots x_n$, are *coupled* by the n^2 equations (124) or (125). As an example, consider the electro-magnetically coupled currents J_1, J_2 and voltages V_1, V_2 of two linked electric oscillating circuits (Fig. 28), satisfying the differential equations

$$\left.\begin{aligned} &L_1\frac{dJ_1}{dt}+M\frac{dJ_2}{dt}-V_1=0\\ &L_2\frac{dJ_2}{dt}+M\frac{dJ_1}{dt}+V_2=0\\ &C_1\frac{dV_1}{dt}+J_1=0\\ &C_2\frac{dV_2}{dt}-J_2=0\end{aligned}\right\} \tag{127}$$

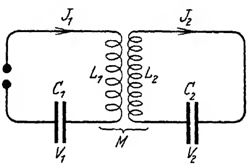


Fig. 28. Two coupled circuits.

where C_1, C_2 designate the capacitances, L_1, L_2 the inductances of the respective circuits and M the mutual inductance. In the above system of equations (127), comparing with equation (124) and substituting $x_1 = J_1$, $x_2 = J_2$, $x_3 = V_1$, $x_4 = V_2$, we have:

$$\begin{array}{llll} L_{11}(D) = L_1 D, & L_{12}(D) = MD, & L_{13}(D) = -1, & L_{14}(D) = 0, \\ L_{21}(D) = MD, & L_{22}(D) = L_2 D, & L_{23}(D) = 0, & L_{24}(D) = +1, \\ L_{31}(D) = +1, & L_{32}(D) = 0, & L_{33}(D) = C_1 D, & L_{34}(D) = 0, \\ L_{41}(D) = 0, & L_{42}(D) = -1, & L_{43}(D) = 0, & L_{44}(D) = C_2 D. \end{array}$$

6. Linear Homogeneous Systems with Fixed Coefficients.

Let us now assume that the coefficients $a_{jk,\lambda}$ are all independent of t and that the presented equations are homogeneous (equation (123)). The system then is:

$$\sum_{k=1} L_{jk}(D) x_k = 0 \qquad (j = 1, 2 \cdots n) \tag{128}$$

The solution for this system shall be given without proof; it is

$$x_k(t) = P_{1k}(t) e^{r_1 t} + P_{2k}(t) e^{r_2 t} + \cdots, \tag{129}$$

where $r_1, r_2 \cdots$ (the so-called "characteristic values"* of the problem) represent the roots of the algebraic equation (Principal equation)

$$\Delta(r) = \begin{vmatrix} L_{11}(r) L_{12}(r) \cdots L_{1n}(r) \\ L_{21}(r) L_{22}(r) \cdots L_{2n}(r) \\ \\ L_{n1}(r) L_{n2}(r) \cdots L_{nn}(r) \end{vmatrix} = 0 \tag{130}$$

and the functions $P_{1k}(t)$, $P_{2k}(t)$, $\cdots$ are polynomials of t (with arbitrary fixed coefficients) the degree of which is smaller by 1 than the multiplicity of the corresponding root $r_1, r_2 \cdots$ of the principal equation (130). This solution, equation (129), is, it will be recognized, an obvious generalization of the case of one single linear differential equation

$$L(D)x = 0$$

with constant coefficients. Now, system (127) (belonging to the homogeneous equations) should be treated in the above manner.

7. Linear Systems with Constant Coefficients; Systems Excited by Harmonic Oscillations.

Let us again consider the system (125), having this time constant coefficients $a_{jk,\lambda}$, and let us assume that the exciting accelerations $T_j(t)$

* Translator's remark: the German word is *Eigenwert*, a word frequently used in English textbooks as well.

are harmonic oscillations. In this case, instead of starting from sine or cosine functions, one may represent them by exponential functions in the complex plane. Thus, we may put

$$T_j(t)=A_je^{pt}(j=1,\ 2,\cdots n) \tag{131}$$

where $A_1, A_2, \cdots A_n$ designate real constants and p a complex constant. The latter determines the mutual frequency and in certain cases, namely if

$$\operatorname{Re} p\leq 0, \tag{132}$$

also the mutual damping constant of all exciting oscillations. We realize this at once if we put $p=\alpha+i\omega$ and if we write $T_j(t)$ in the form:

$$T_j(t)=A_je^{\alpha t}(\cos\omega t+i\sin\omega t).$$

The quantities $A_1, A_2, \cdots A_n$ designate amplitudes. Let us thus consider the system

$$\sum_{k=1}^{n} L_{jk}(D)x_k=A_je^{pt} \tag{133}$$
$$(j=1,\ 2,\cdots n)$$

and let us restrict ourselves to the case of the complex constant p *not* belonging to the characteristic values of the problem, i.e. to the case when the determinant reads as follows:

$$\Delta(p)=\begin{vmatrix} L_{11}(p)L_{12}(p)\cdots L_{1n}(p)\\ L_{21}(p)L_{22}(p)\cdots L_{2n}(p)\\ \\ L_{n1}(p)L_{n2}(p)\cdots L_{nn}(p)\end{vmatrix}\neq 0. \tag{134}$$

In order to integrate the system (133) note that it is sufficient to find a system of particular solutions; for, by equation (123), we only have to integrate the corresponding homogeneous system, the general solution of which (equation (129)) is known. Referring to the system (133) we may start by writing

$$x_k(t)=C_k(p)e^{pt} \qquad (k=1,\ 2,\cdots n), \tag{135}$$

where the $C_k(p)$ designate still undetermined constants (i.e. quantities independent of t). Equation (135) satisfies equations (133) if only the conditions

$$\sum_{k=1}^{n} L_{jk}(p)C_k(p)=A_j \tag{136}$$
$$(j=1,\ 2,\cdots n)$$

are met, as may be shown by a simple calculation. Equation (136) comprises n linear equations with the unknowns $C_k(p)$ $(k=1,\ 2,\cdots n)$; by

assumption (equation (134)), i.e. that $\Delta(p) \neq 0$, we may uniquely compute these unknowns from the above n equations. The solution gives

$$C_k(p) = \frac{\Delta_k(p)}{\Delta(p)}, \tag{137}$$

where $\Delta_k(p)$ signifies the determinant obtained from $\Delta(p)$ by substitution of the fixed values $A_1, A_2, \cdots A_n$ of the amplitudes in place of the k^{th} column of said determinant. Thus, the required solution of the system (133) is:

$$x_k(t) = \frac{\Delta_k(p)}{\Delta(p)} e^{pt} \tag{138}$$

$$(k = 1, 2, \cdots n).$$

8. Linear Systems with Constant Coefficients; Systems Excited by Impulse Functions.

The discussions of the last articles should, in the major part, be considered as preparations for the investigations now to follow; these investigations shall again illustrate the methods of function theory.

Let then the exciting accelerations be impulse functions. Such systems of oscillations, excited by impulse functions, especially appear in switching problems of electrical engineering, if, as for instance when telegraphing, the closing of a switch produces oscillations of current and voltage in the line. Using the impulse function $S(t)$ as indicated in equation (117), we represent the exciting functions of the differential equations as follows:

$$T_j(t) = A_j S(t) = \frac{A_j}{2\pi i} \int e^{tp} \frac{dp}{p}. \tag{139}$$

$$(j = 1, 2, \cdots n)$$

The complex variable of integration is now denoted by p and the hook-integrals are now taken in the complex p-plane. The constants A_j $(j = 1, 2, \cdots n)$ now have the meaning of intensities of the individual current impulses. Thus, the differential equations are

$$\sum_{k=1}^{n} L_{jk}(D) x_k = A_j S(t). \tag{140}$$

$$(j = 1, 2, \cdots n)$$

To integrate, i.e. to obtain a system of particular solutions, we start as follows:

$$x_k(t) = \frac{1}{2\pi i} \int e^{pt} C_k(p) \frac{dp}{p} \tag{141}$$

$$(k = 1, 2, \cdots n)$$

where $C_1(p)$, $C_2(p)$ designate analytic functions (as yet unknown) of the complex variable p, these functions, however, satisfying certain conditions. We have to refrain from formulating them here in detail. We shall only mention that the integrals appearing in equation (141) must not only be all existent, but it must also be possible to differentiate the functions represented by above integrals as many times with respect to t as the differential operators $L_{jk}(D)$ demand it, and moreover, it *must be possible to carry out the differentiation with respect to t under the integral sign.* The last condition especially requires an exact investigation, the discussion of which would go beyond the scope of this book. Referring to a restricted case we shall later convince ourselves of the admissibility of the above assumption.

From these assumptions and from equation (141) we deduce that

$$\frac{d^\lambda x_k}{dt^\lambda} = D^\lambda x_k = \frac{1}{2\pi i}\int p^\lambda e^{pt} C_k(p)\,\frac{dp}{p} \tag{142}$$

and, by equation (126), or

$$L_{jk}(D) = \sum_\lambda a_{jk,\lambda}\cdot D^\lambda, \qquad (\lambda=0,\ 1,\ 2,\cdots m_{jk}) \tag{143}$$

that

$$\begin{aligned} L_{jk}(D)x_k &= \frac{1}{2\pi i}\sum a_{jk,\lambda}\int p^\lambda e^{pt} C_k(p)\,\frac{dp}{p} \\ &= \frac{1}{2\pi i}\int\Big(\sum_\lambda a_{\lambda j,\lambda} p^\lambda\Big) e^{pt} C_k(p)\,\frac{dp}{p}. \end{aligned}$$

Using equation (143) we have:

$$L_{jk}(D)x_k = \frac{1}{2\pi i}\int L_{jk}(p)e^{pt}C_k(p)\,\frac{dp}{p} \tag{144}$$

$$(j,\ k=1,\ 2,\cdots n).$$

Substitute this result for the left side of equation (140) and at the same time note that the right sides — by virtue of equation (139) — may also be written in form of hook-integrals; consequently, we get

$$\int\Big(\sum_{k=1}^{n} L_{jk}(p)C_k(p) - A_j\Big)e^{tp}\,\frac{dp}{p} = 0 \tag{145}$$

$$(j=1,\ 2,\cdots n),$$

and these n equations are satisfied identically if the unknown functions $C_k(p)$ are determined in such a way as to comply with the conditions:

$$\sum_{k=1}^{n} L_{jk}(p)C_k(p) = A_j$$

$$(j=1,\ 2,\cdots n).$$

But these conditions literally agree with equations (136) and yield as before:

$$C_k(p) = \frac{\Delta_k(p)}{\Delta(p)}. \tag{137}$$

Referring to the explanations of $\Delta(p)$ and $\Delta_k(p)$ — compare Art. 7 — these determinants are integer rational functions of p and thus $C_1(p)$, $C_2(p), \cdots C_n(p)$ are *fractional rational functions of* p. Substituting them in equation (141), the so obtained functions $x_k(t)$ represent the required solutions of the system (140), *bearing in mind the above mentioned assumptions.*

9. Heaviside's and K. W. Wagner's Formula.

Let

$$x(t) = \frac{1}{2\pi i}\int_{\circlearrowleft} e^{pt} C(p)\,\frac{dp}{p} \tag{146}$$

be any one of the previously considered functions, for example a function satisfying the system of differential equations (140). Starting out from such a formula, at which he arrived by physical reasoning, K. W. Wagner* proved an empirical† formula by Heaviside.

In order to join the Heaviside-Wagner notation, put

$$C(p) = \frac{1}{Z(p)} \tag{147}$$

The function $Z(p)$ is called the "characteristic function" of $x(t)$, the equation

$$Z(p) = 0 \tag{148}$$

is called the "characteristic equation." The roots of this equation correspond to the characteristic values of the problem, i.e. to the poles of $C(p)$, if $C(p)$ is a rational function of p. Let us consider the equation

$$pZ(p) = 0; \tag{148a}$$

the roots are

$$p_0 = 0, \quad p_1, p_2, \cdots p_\nu$$

with the numbers of order

$$k_0, k_1, k_2, \cdots k_\nu.$$

Heaviside's formula for $x(t)$, supplemented by Wagner, reads

$$x(t) = \sum_\nu Z_\nu(t), \tag{149}$$

* *Arch. Elektrot,* Vol. 4 (1916), p. 159.

† Mathematics is an experimental science.

where ν assumes the values 0, 1, 2, $\cdots$ In the above formula

$$Z_\nu(t) = e^{tp_\nu}\left(A_{\nu 1} + A_{\nu 2}\frac{t}{1!} + A_{\nu 3}\frac{t^2}{2!} + \cdots A_{\nu k_\nu}\frac{t^{k_\nu - 1}}{(k_\nu - 1)!}\right), \quad (150)$$

where $A_{\nu 1}, A_{\nu 2} \cdots A_{\nu k_\nu}$ are the coefficients of the principal part (compare Sect. E, Art. 5) of Laurent's expansion of

$$\frac{C(p)}{p} = \frac{1}{pZ(p)}$$

in powers of $(p - p_\nu)$, i.e. it is

$$\frac{C(p)}{p} = \frac{1}{pZ(p)} = \frac{A_{\nu, k_\nu}}{(p-p_\nu)^{k_\nu}} + \frac{A_{\nu, k_\nu - 1}}{(p-p_\nu)^{k_\nu - 1}} + \cdots + \frac{A_{\nu 1}}{p - p_\nu} + \mathrm{P}(p - p_\nu), \quad (151)$$

where $\mathrm{P}\,(p - p_\nu)$ is understood to be a power series progressing in ascending powers of $(p - p_\nu)$.

If equation (148a) yields only *ordinary* roots $p_0 = 0$, p_1, p_2, $\cdots$, the quantity $Z_\nu(t)$ reduces (for $p_\nu \neq 0$) to

$$Z_\nu(t) = \frac{e^{tp_\nu}}{p_\nu \cdot Z'(p_\nu)} \qquad (\nu = 1, 2, \cdots) \qquad (152)$$

where $Z'(p) = \dfrac{dZ}{dp}\cdot$ That is to say, in this case equations (150) and (151) simplify to

$$Z_\nu(t) = e^{tp_\nu} A_{\nu 1}, \quad (153)$$

$$\frac{1}{pZ(p)} = \frac{A_{\nu 1}}{p - p_\nu} + \mathrm{P}(p - p_\nu). \quad (154)$$

From the last equation it follows that

$$A_{\nu 1} = \frac{1}{p}\frac{p - p_\nu}{Z(p)} - (p - p_\nu)\mathrm{P}(p - p_\nu),$$

the last term vanishing for $p \to p_\nu$ and the first one, on account of $Z(p_\nu) = 0$, becoming

$$\frac{1}{p_\nu} \cdot \frac{1}{Z'(p_\nu)},$$

in accordance with the Bernouilli-L'Hospital rule; p_ν being an ordinary

root, we have $Z'(p_\nu) \neq 0$. Substituting this expression for $A_{\nu 1}$ in equation (153), we thus obtain equation (152).

If, on the contrary, $p_\nu = 0$, $\nu = 0$ formulas (153) and (154) still hold, but now

$$A_{01} = \frac{1}{Z(0)},$$

where $Z(0) \neq 0$ on account of the assumption that the root $p_0 = 0$ is an ordinary root. Hence, $Z_0(t) = \dfrac{1}{Z(0)}$. Substituting everything back in equation (149), we obtain,

$$x(t) = \frac{1}{Z(0)} + \sum_\nu \frac{e^{tp_\nu}}{p_\nu Z'(p_\nu)} \qquad (\nu = 1, 2, \cdots) \tag{155}$$

Formula (155) represents the case originally considered by Heaviside.*

10. Proof of the Heaviside-Wagner Formula.

To prove this formula the following four assumptions shall be made:

1. $Z(p)$ is one-valued.

2. The characteristic values $p_\nu (\nu = 1, 2, \cdots)$ do not possess positive real components.

3. $C(p) = \dfrac{1}{Z(p)}$ does not become arbitrarily large for all $|p| > \Omega$; there should, rather, always be values $R > \Omega$ such that for all $|p| = R$ the function $C(p)$ remains restricted.

4. $Z(p)$ possesses only isolated zeros $C(p)$, therefore, only isolated poles but no essential singular points; $C(p)$ thus is a meromorphic function (compare Sect. E, Art. 5).

Consider now the integral of equation (146)

$$J = \frac{1}{2\pi i} \oint e^{tp} C(p) \frac{dp}{p} = \frac{1}{2\pi i} \oint \frac{e^{tp} dp}{p \cdot Z(p)} \tag{156}$$

extended over the closed path of integration of the complex p-plane as indicated in Fig. 29. By the residue theorem

$$J = \sum_{\nu=0}^{n} \operatorname{Res}\left(\frac{e^{tp}}{p \cdot Z(p)}\right)_{p=p_\nu}, \tag{157}$$

* Translator's remark: This formula is called Heaviside's Expansion Theorem in the English literature.

where $p_0 = 0, p_1, p_2, \cdots p_n$ denote the poles inside of the path of integration; by assumption, these poles can only appear there in a finite number. Let the notation be chosen in such a way that $|p_\nu| \leq |p_{\nu+1}|$ and, if $|p_\nu| = |p_{\nu+1}|$, that arc $(p_\nu) <$ arc $(p_{\nu+1})$. Let k_ν be the number of order of the solution p_ν of equation $pZ(p) = 0$, or k_ν the number of order of the pole p_ν of the function $\frac{1}{pZ(p)}$. Then, Laurent's expansion in powers of $\zeta = p - p_\nu$, equation (151), holds. On the other hand, we have

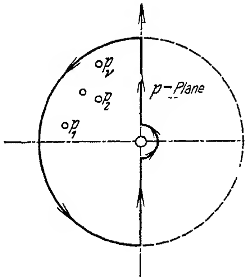


Fig. 29. Referring to Heaviside-Wagner Formula.

$$e^{tp} = e^{tp_\nu} \quad e^{t\zeta} = e^{tp_\nu}\left(1 + \frac{t\zeta}{1!} + \frac{t^2\zeta^2}{2!} + \cdots\right)$$

and thus

$$\begin{aligned} \frac{e^{tp}}{pZ(p)} = e^{tp_\nu}\Big(1 + \frac{t}{1!}\zeta + \frac{t^2}{2!}\zeta^2 + \cdots \\ \cdot\left(\frac{A_{\nu,k_\nu}}{\zeta^{k_\nu}} + \frac{A_{\nu,k_\nu-1}}{\zeta^{k_\nu-1}} + \cdots + \frac{A_{\nu,1}}{\zeta} + \mathrm{P}(\zeta)\right)\Big) \end{aligned} \tag{158}$$

From this formula the coefficient of $\zeta^{-1} = (p - p_\nu)^{-1}$ may be easily obtained, viz.

$$\begin{aligned} &\operatorname{Res}\left(\frac{e^{tp}}{pZ(p)}\right)_{p=p_\nu} \\ &= e^{tp_\nu}\left(A_{\nu 1} + A_{\nu 2}\frac{t}{1!} + A_{\nu 3}\frac{t^2}{2!} + \cdots + A_{\nu,k_\nu}\frac{t^{k_\nu-1}}{(k_\nu-1)!}\right) \end{aligned} \tag{159}$$

This expression agrees with the right side of equation (150); consequently, by equation (157):

$$J = \sum_{\nu=0} Z_\nu(t) \tag{160}$$

If we now let the radius R in Fig. 29 grow unlimitedly, however in such a way as to comply with assumption 3, the number of characteristic values p_ν — if this number is finite — (by reason of assumptions 2 and 4) shall all be bounded by the semi-circle (starting from a certain R-value); or else, if there exists an infinite number of characteristic values p_ν, each one ultimately may be bounded by the semi-circle.

On account of $C(p)$ being restricted, and considering the results of Art. 1 and of the following ones, integral (156), for $R \rightarrow \infty$, goes over in the corresponding hook-integral (146), while the residue sum (160) either represents a finite series with as many terms as there exist different characteristic values, or an infinite series. This proves the Heaviside-Wagner formula (149). The convergence of the infinite series should, of course, be demonstrated for every individual case.

11. Supplementary Notes to Art. 8.

If the number m of the characteristic values p_ν is finite, $C(p)$ is an integer rational function. By equations (146) and (149) we then have:

$$x(t) = \frac{1}{2\pi i}\int e^{pt}C(p)\,\frac{dp}{p} = \sum_{\nu=0}^{m} Z_\nu(t). \tag{161}$$

We have recognized such functions to be solutions of linear differential equations, the disturbance-terms of which are impulse functions. It remains to be proved that the differentiation with respect to the argument t is permitted under the integral sign. But this may be easily demonstrated in the case of equation (161). Differentiating under the integral sign, $pC(p) = C(p)_1$ takes the place of $C(p)$ and also $p^{-1}Z(p) = Z(p)_1$ the place of $Z(p)$. Let us then expand the so obtained integral using the residue theorem, thus putting

$$\frac{1}{2\pi i}\int e^{pt}C(p)_1\,\frac{dp}{p} = \sum_{\nu=0}^{m} Z_\nu(t)_1. \tag{162}$$

From the assumptions of Art. 10, assumptions 1, 2 and 4 apply without further comments to $C(p)_1$ and $Z(p)_1$ as well; also, the number m of the different poles remains the same, with the exception that the pole $p=0$ might possibly not appear at all. The assumption 3 alone should be explicitly maintained for $C(p)_1$. Then:

$$Z_\nu(t)_1 = \mathrm{Res}\left(\frac{e^{tp}}{pZ(p)_1}\right)_{p=p_\nu}.$$

By equation (158) and introducing $p = \zeta + p_\nu$:

$$\frac{e^{tp}}{pZ(p)_1} = \frac{(\zeta+p_\nu)e^{tp}}{pZ(p)} = \frac{\zeta e^{tp}}{pZ(p)} + p_\nu\frac{e^{tp}}{pZ(p)}$$

$$= e^{tp_\nu}\left(\zeta + \frac{t}{1!}\zeta^2 + \frac{t^2}{2!}\zeta^3 + \cdots\right)$$

$$\cdot\left(\frac{A_{\nu}k_\nu}{\zeta^{k_\nu}} + \frac{A_{\nu,k_\nu-1}}{\zeta^{k_\nu-1}} + \cdots + \frac{A_{\nu 1}}{\zeta} + \mathrm{P}(\zeta)\right).$$

From this we obtain the residue as the coefficients of ζ^{-1} as follows:

$$Z_\nu(t) = e^{tp_\nu}\left[A_{\nu 2}+A_{\nu 3}\frac{t}{1!}+\cdot \quad + A_\nu \frac{t^{k_\nu-2}}{(k_\nu-2)!}\right.$$
$$\left.+p_\nu\left(A_{\nu 1}+A_{\nu 2}\frac{t}{1!}+\ldots+A_{\nu k_\nu}\frac{t^{k_\nu-1}}{(k_\nu-1)!}\right)\right]. \tag{163}$$

On the other hand, differentiation of the right side of equation (161) gives

$$\frac{dx(t)}{dt} = Dx(t) = \sum_{\nu=0} Z'_\nu(t) \tag{164}$$

and it may easily be shown by differentiation of equation (150) that

$$Z'_\nu(t) = Z_\nu(t)_1 \tag{165}$$

From this we follow the identity of equations (164) and (162) and, consequently, the *validity of differentiation of the hook-integral* (161) *under the integral sign.*

In equation (164) the characteristic value $p=0$ would not appear any more if $Z'_0(t)$ would be identically equal zero, i.e. if $Z_0(t)$ would be constant. This, as indicated by formula (151), is the case if, and only if, $C(p)$ is regular for $p=0$. But then the function $pC(p)=C(p)_1$ is no longer restricted for arbitrarily large values of $|p|$, and thus assumption 3 in Art. 10 is no longer complied with. This case, therefore, has to be excluded.

If we differentiate $x(t)$ multiply with respect to t (under the integral sign), the same considerations hold; only, differentiating λ-times, we should explicitly assume that $p^\lambda C(p)$, even for arbitrarily large values of $|p|$, is still restricted. In the rational function $C(p)$ itself the pole $p=0$ must thus at least possess the number of order λ.

We now proceed to the most important geometric application of complex functions, the conformal mapping.

12. Conformal Mapping. Let x, y (imagined as functions of a parameter) represent the coördinates of a variable point of a curve. The vector

$$dz = dx + idy = ds \cdot e^{i\vartheta} \tag{166}$$

runs tangentially to the curve because of

$$\vartheta = \operatorname{arc}(dz) = \tan^{-1}\frac{dy}{dx}.$$

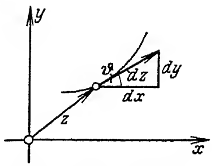


Fig. 30. Vector of the line-element.

The length of the vector $|dz|=\sqrt{dx^2+dy^2}=ds$ is the arc-element of the curve (Fig. 30). Let $w=f(z)$ be an analytic function of z. Then,

$$dw=f'(z)dz. \tag{167}$$

By means of $w=f(z)$, a relation between the points z of the z-plane and the corresponding points w of the w-plane is established, or — as one says — the planes are mapped one upon the other. The curves (I) and (II) of the z-plane correspond to the curves (I) and (II) of the w-plane (Fig. 31).

Put

$$|dw|=dS;\ \text{arc}\ (dw)=\Theta$$

so that

$$dw=dS\cdot e^{i\Theta}. \tag{168}$$

From equation (167) we have:

$$dS\cdot e^{i\Theta}=f'(z)ds\cdot e^{i\vartheta} \tag{169}$$

In this equation we now may compare arguments and moduli, if only $f(z)\neq 0$. The comparison of the moduli yields

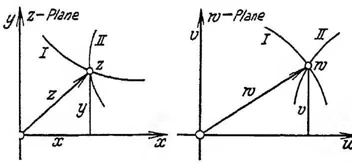


Fig. 31. Referring to conformal mapping.

$$dS=f'(z)ds \tag{170}$$

and that of the arguments

$$\Theta=\text{arc}\,f'(z)+\vartheta. \tag{171}$$

Hence, $f'(z)$ measures the ratio of the lengths of corresponding line-elements, while arc $f'(z)$ designates the rotation which took place in the process of mapping a line-element.* Both only depend upon the locus z of the z-plane to be mapped. Thus, if (I) and (II) are two curves of the z-plane going through the point z, and ds_1, ds_2 are their line-elements, ϑ_1, ϑ_2 the corresponding angles of their tangents, then, by equations (170) and (171),

$$dS_1: dS_2=ds_1: ds_2 \tag{172}$$

and

$$\Theta_1-\Theta_2=\vartheta_1-\vartheta_2\,. \tag{173}$$

These equations state that the ratios of lengths are the more preserved the smaller the lengths are and that the angles remain unchanged. The mapping of the z-plane upon the w-plane, transacted by means of an *analytic* function $w=f(z)$, is, therefore, said to be *similar in the smallest*

* We, therefore, use to call $f'(z)$ the "ratio of distortion."

parts or *conformal* and *isogonal.* All these considerations hold, as already mentioned, only *under the assumption that* $f'(z)\neq 0$.

13. Examples of Conformal Mapping.

A very simple confirmation of the fact that in conformal mapping the angles are preserved is furnished by the Cauchy-Riemann differential equations (11). For, these equations yield at once

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0, \tag{174}$$

this, as is well known, being the condition for rectangular intersection of the families of curves $u=\text{const}$ and $v=\text{const}$ of the z-plane. The mappings of these two families of curves in the w-plane ($w=u+iv$) are simply the straight lines $u=\text{const}$ and $v=\text{const}$, i.e. the coördinates.

Let us now discuss briefly some of the simplest conformal mappings.

1. Let

$$w=z-z_0.$$

This mapping represents a shifting of the z-plane along the vector z_0.

2. $w=mz$, for a real $m>0$, results in a change of scale of the z-plane.

3. $w=az$, for a complex a, represents rotation and extension, for we have

$$|w|=|a|\cdot|z|;\quad \text{arc }(w)=\text{arc }(a)+\text{arc }(z).$$

The first equation means change in length, the second — rotation.

4. $w=az+b$ (a and b being constant) furnishes a combination of 1. and 3., thus, a shift along b, a change in scale in the ratio $|a|$ after rotation through arc (a).

5. $w=\dfrac{1}{z}$ represents a transformation by means of reciprocal radii with a following reflection at the x-axis; for we have

$$|w|=\frac{1}{|z|};\ \text{arc }(w)=-\text{arc }(z).$$

Herewith, the origin $z=0$ moves into infinity ($w=\infty$).

14. Continuation: Linear Fractional Functions.

The linear fractional transformation

$$w=\frac{\alpha z+\beta}{\gamma z+\delta}, \tag{175}$$

where α, β, γ, δ are four complex constants subjected to the condition

$\alpha\delta-\beta\gamma\neq 0$, may be traced back to the conformal mappings 4 and 5 of the preceding Article by means of the transformation

$$w\equiv\frac{\alpha}{\gamma}-\frac{\alpha\delta-\beta\gamma}{\gamma(\gamma z+\delta)}=A+\frac{B}{\zeta},$$
$$\zeta=\gamma z+\delta.$$

First, use the conformal mapping of the z-plane upon the ζ-plane, which requires for the z-plane (according to Art. 13, 4) a change in scale in the ratio $|\gamma|$, a rotation through the angle arc (γ) and then a shift along the vector δ. Then, map the ζ-plane upon the w-plane, which operation is achieved if one performs a transformation by means of reciprocal radii and a reflection at the real axis, if one then adds a rotation through arc $(B)=$ arc $(\gamma)-$arc $(\alpha\delta-\beta\gamma)$ as well as a change in scale in the ratio $|B|=|\alpha\delta-\beta\gamma|:|\gamma|$ and if one finally shifts the ζ-plane along the vector $A=\dfrac{\alpha}{\gamma}$. Herewith, $\gamma\neq 0$ has been tacitly understood. For $\gamma=0$, equation (175) would be of the form 4 (in the previous Article). Furthermore,

$$\frac{dw}{dz}=\frac{\alpha\delta-\beta\gamma}{(\gamma z+\delta)^2};$$

hence it becomes apparent why the assumption $\alpha\delta-\beta\gamma\neq 0$ had to be made.

By means of the conformal mapping (175), the totality of all circles (and straight lines) of the z-plane is transferred into the totality of all circles (and straight lines) of the w-plane. In this connection, the mapping of three points of the w-plane may be arbitrarily prescribed, corresponding to the number of the available constants α, β, γ, δ.

(a) In order to study the special linear fractional transformation

$$w=\frac{\alpha z+\beta}{\gamma z+\delta},$$

where now α, β, γ, δ are assumed to be reals and $\alpha\delta-\beta\gamma>0$, put $w=u+iv$; the conjugate value is

$$\tilde{w}=\frac{\alpha\tilde{z}+\beta}{\gamma\tilde{z}+\delta}=u-iv.$$

Therefore,

$$\frac{w-\tilde{w}}{2i}=v=\frac{(\alpha\delta-\beta\gamma)}{(\gamma z+\delta)\cdot(\gamma\tilde{z}+\delta)}\,y.$$

As, from the above, v differs from y by a *positive real* factor, the just considered function w transacts a mapping of the upper half-plane $y\geqq 0$ upon the upper half-plane $v\geqq 0$.

(b) Consider the linear fractional transformation

$$w = \frac{i-z}{i+z}.$$

Here,

$$|w| = \left|\frac{i-z}{i+z}\right| = \sqrt{\frac{x^2+(y-1)^2}{x^2+(y+1)^2}} = \sqrt{\frac{x^2+y^2+1-2y}{x^2+y^2+1+2y}}.$$

For $y=0$, we obtain

$$|w|=1.$$

The x-axis is transferred into the unit circle of the w-plane. For $y>0$,

$$|w|<1,$$

i.e. the upper half-plane $y>0$ goes over into the *inside* of the unit circle $|w|=1$. Into which curve of the w-plane does the circle $|z|=1$ go over?

15. Mappings of Riemann Surfaces.

The mappings which we have considered till now were all transacted by means of one-valued (i.e. one-valued with one-valued inverse) functions. Consequently, every point of the z-plane corresponded to exactly one point of the w-plane and vice versa. This is no longer the case with the following mappings. Therefore, the mappings of the whole z-plane may now be part regions of the w-plane, or may also multiply overlap it. Two examples for this case are the following ones:

1. $w=z^n$ for $n>0$, integer.

The inverse function

$$z=\sqrt[n]{w}$$

is n-valued and becomes one-valued on a n-times overlapped w-plane — the corresponding Riemann surface. The function thus maps this Riemann surface reversibly and uniquely upon the smooth z-plane (*schlichte Ebene*).

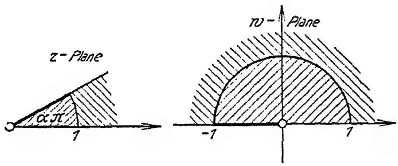


Fig. 32. Conformal mapping of an angular space upon a half-plane.

2. The function $w=z^{1/a}$, where a is real, maps a sector of the unit circle with the angle $a\pi$ upon the semi-circle of the w-plane (Fig. 32).

16. Conformal Mappings of a Convex Polygon upon a Half-Plane.

Let us next discuss a particularly important mapping, first treated by Christoffel (1867) and by H. A. Schwarz (1869), starting out with the function

$$w=f(z)=\int^{z} \frac{dz}{(z-x_1)^{a_1}(z-x_2)^{a_2}\cdots(z-x_n)^{a_n}}. \tag{176}$$

Let $x_1<x_2<\cdots x_n$ be arbitrary real values and $\alpha_1, \alpha_2, \cdots \alpha_n$ real constants, the latter ones being subjected to the conditions

$$0<\alpha_k<1 \qquad (k=1, 2, \cdots n) \tag{177}$$

and

$$\alpha_1+\alpha_2+\cdots+\alpha_n=2. \tag{178}$$

Let the variable z be restricted to the upper half-plane Im $z\geqq 0$. The path of integration in (176) may start at an arbitrary point z_0 of the upper half-plane and may take an arbitrary course in that plane; it must, however, avoid the points $x_1, x_2 \cdots x_n$. In order to confer a one-valued significance to the function $w=f(z)$, $(z-x_k)^{\alpha_k}$ is understood to be the following value:

$$(z-x_k)^{\alpha_k}=|z-x_k|^{\alpha_k}\cdot e^{i\alpha_k \operatorname{arc}(z-x_k)}, \qquad \text{where } 0< \operatorname{arc}(z-x_k)<\pi. \tag{179}$$

Then, w is a one-valued and analytic function in the upper half-plane, for it has the derivative

$$\frac{dw}{dz}=\frac{1}{(z-x_1)^{\alpha_1}(z-x_2)^{\alpha_2}\cdots(z-x_n)^{\alpha_n}}=f'(z), \tag{180}$$

which exists for any value of $z\neq x_1, x_2, \cdots x_n$ and vanishes nowhere, this also being the case for $z=\infty$, as the substitution $\zeta=\dfrac{1}{z}$ shows. Now,

$$\operatorname{arc}\bigl(f'(z)\bigr)=-\alpha_1 \operatorname{arc}(z-x_1)-\alpha_2 \operatorname{arc}(z-x_2)-\cdots-\alpha_n \operatorname{arc}(z-x_n). \tag{181}$$

If the point z travels along the x-axis in the positive sense, we have in the region $-\infty<z<x_1$

$$\operatorname{arc}(z-x_1)=\pi,\ \operatorname{arc}(z-x_2)=\pi, \cdots \operatorname{arc}(z-x_n)=\pi,$$

while for $x_1<z<x_2$ we get

$$\operatorname{arc}(z-x_1)=0,\ \operatorname{arc}(z-x_2)=\pi, \cdots \operatorname{arc}(z-x_n)=\pi.$$

By means of a small semi-circle *in the upper half-plane* Im $z\geqq 0$, we may avoid the point $z=x_1$. A correspondent behavior occurs at the remaining points $x_2, x_3, \cdots x_n$, and we easily realize that $\operatorname{arc}\bigl(f'(z)\bigr)$ preserves a constant value on each of the individual distances $-\infty \cdots x_1, x_1 \cdots x_2, x_2 \cdots x_3, \cdots x_{n-1} \cdots x_n, x_n \cdots +\infty$. According to formula (171), where ϑ is zero, also Θ has a constant value. Any of the above distances of the x-axis, therefore, correspond to a line section in the w-plane and,

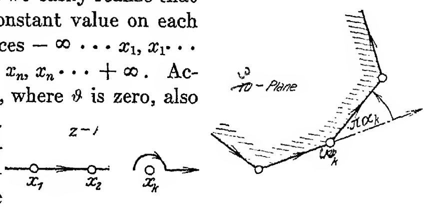


Fig. 33. Conformal mapping of a half-plane upon a polygon.

because of the continuity of $f(z)$, w describes in the w-plane a polygon, while z travels along the x-axis from $-\infty$ to $+\infty$. In the neighborhood of $z=x_k$ the arcus of $(z-x_k)$ reduces from π to 0, the change in arc $(f'(x))$ thus being $(-a_k)\cdot(-\pi)=a_k\pi$. This is the outside angle of the polygon in the w-plane (Fig. 33) at the corresponding corner $w=w_k$. The sum of these outside angles is, because of equation (178),

$$\Sigma a_k\pi=2\pi.$$

The function w otherwise being regular (also at $z=\infty$), the polygon contains exactly the n corners $w_1, w_2, \ldots w_n$; it is convex and closed. Thus, the function given by equation (176) transacts, under the conditions (177), (178) and (179), a conformal mapping of the real axis of the z-plane upon the boundary of a convex and closed polygon of the w-plane. From equation (180) it is evident that the mapping at the corners of the polygon is no longer a conformal one. We have, it is true, not yet proved that the inside of the polygon is mapped upon the upper half-plane Im $z>0$ as well.

As an illustration of the discussed conformal mapping, assume $n=4$ and

$$a_1=a_2=a_3=a_4=\frac{1}{2}.$$

The resulting polygon is a rectangle and the mapping function, by which this rectangle of the w-plane is conformally mapped upon the real axis of the z-plane,

$$w=\int_{z_0}^{z}\frac{dz}{\sqrt{(z-x_1)(z-x_2)(z-x_3)(z-x_4)}} \tag{182}$$

is an elliptic integral of first kind.

17. Conformal Mapping of a Convex Polygon upon the Unit Circle.

To solve this problem we only have to map conformally the z-plane of the preceding Article upon a ζ-plane in such a way that the upper half of the z-plane corresponds to the unit circle of the ζ-plane. This problem has already been solved in Art. 14, b. Put, therefore,

$$\zeta=\frac{i-z}{i+z}$$

i.e.

$$z=i\,\frac{1-\zeta}{1+\zeta}.$$

Let the points $x_1, x_2, \ldots x_n$ of the x-plane correspond to the points $\zeta_1, \zeta_2, \ldots \zeta_n$ of the ζ-plane, the latter points, incidentally, being located on the unit circle (compare Art. 14, b). Then,

$$z-x_k=\frac{-2i}{1+\zeta_k}\frac{\zeta-\zeta_k}{1+\zeta}$$

$$dz=\frac{-2i}{1+\zeta_k}\cdot\frac{d\zeta}{(1+\zeta)^2}.$$

Substituting this in equation (176) we obtain after a few intermediate calculations

$$w=C\int_{\zeta_0}^{\zeta}\frac{d\zeta}{(\zeta-\zeta_1)^{\alpha_1}(\zeta-\zeta_2)^{\alpha_2}\ldots(\zeta-\zeta_n)^{\alpha_n}}, \tag{183}$$

where

$$C=i(1+\zeta_1)^{\alpha_1}(1+\zeta_2)^{\alpha_2}\ldots(1+\zeta_n)^{\alpha_n}$$

$$\zeta_0=\frac{1-z_0}{1+z_0}.$$

Formula (183), seemingly of the same form as formula (176), thus transacts a conformal mapping of a polygon of the w-plane upon the periphery of the unit circle of the ζ-plane. In addition to the formerly stated conditions, we now have the conditions:

$$|\zeta_1|=1, \quad |\zeta_2|=1, \ldots |\zeta_n|=1. \tag{184}$$

If, for example,

$$\alpha_1=\alpha_2=\ldots=\alpha_n=\frac{2}{n},$$

we have a regular polygon of n-sides. Choose the points $\zeta_1, \zeta_2, \ldots \zeta_n$ on the unit circle such that they divide the circle into n equal parts, i.e., put

$$\zeta_k=e^{k\frac{2\pi i}{n}} \qquad (k=1, 2, \ldots, n).$$

Then,

$$(\zeta-\zeta_1)(\zeta-\zeta_2)\ldots(\zeta-\zeta_n)=\zeta^n-1,$$

which, if equaled to zero, gives nothing else but the equation of circle division by which $\zeta_k=\sqrt[k]{1}$ is determined. Hence, equation (183) becomes

$$w=C\cdot\int_{\zeta_0}^{\zeta}\frac{d\zeta}{\sqrt[n]{(\zeta^n-1)^2}}. \tag{185}$$

This Abel-integral transacts the conformal mapping of a regular polygon of n sides upon the periphery of the unit circle of the z-plane.

18. Mapping of the Inside of a Polygon.

It only remains to prove that the functions (176) and (183) map also the *inside of the polygon* upon the *upper half-plane* of the z-plane and upon the *inside of the unit circle* of the ζ-plane. Let w_0 be any *inside* point of the polygon. Then, by equation (38),

$$\frac{1}{2\pi i}\oint\frac{d(w-w_0)}{w-w_0}=1$$

where the integral is to be extended over the circumference of the polygon. If we put $w-w_0=F(\zeta)$, this integral, on the other hand, is

$$\frac{1}{2\pi i}\oint\frac{F'(\zeta)}{F(\zeta)}\,d\zeta=1,$$

and we should move once along the complete unit circle of the ζ-plane. But, inside of the unit circle, $F(\zeta)$ is regular everywhere; the integral thus indicates the number of zeros of $F(\zeta)$ inside of the unit circle, by reason of the remark made at the end of Sect. E, Art. 5a. Consequently, $F(\zeta)$ vanishes there exactly once. If ζ_0 represents this zero, then $w=w_0$ at that point. Each inside point w_0 of the polygon, therefore, corresponds to exactly one point ζ_0 inside of the unit circle, which was to be proved. From the above we conclude that by means of function (176) also the inside of the polygon is uniquely and conformally mapped upon the upper half of the z-plane, and vice versa.

19. Schwarz's Reflection Principle.

From the preceding problem of conformal mapping of the circumference and the inside of a polygon in the w-plane upon the real axis and the upper half of the z-plane (by means of the function (176), $w=f(z)$), the question about the conformal mapping of the lower half-plane still remains an open one. The most evident procedure would be to expand the function $w=f(z)$, in the neighborhood of a point sufficiently near to the real axis, in a power series (compare Sect. D, Art. 6) and, if possible, to continue this power series analytically across the x-axis into the lower half-plane. H. A. Schwarz has devised a procedure, in many cases leading more quickly and simply to the goal.

Let (**B**) represent a region of the upper half of the z-plane, bounded by a section AB of the real axis and a curve (C) of the upper half-plane (Fig. 34); let a function $w=f(z)$ (analytic inside and on the boundary of (**B**)) map (**B**) conformally and uniquely upon a region (B) of the w-plane, such that the straight line AB corresponds to a straight line g also in the w-plane, and vice versa. Now, reflect the region (**B**) at

the x-axis, (**B**) going over into (**B′**), and assign the point $\bar{w}$ of the w-plane to the point $\tilde{z}=x-iy$, i.e., to the point of reflection of $z=x+iy$; the point $\bar{w}$ is found by reflection of w at the straight line g. The so produced function

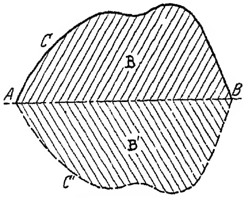


Fig. 34. Referring to Schwarz's reflection principle.

$$\bar{w}=g(\tilde{z})$$

is the analytic continuation of $w=f(z)$ into the lower half-plane. This function maps conformally the reflected region (**B′**) upon the reflection of (B) at the straight line g of the w-plane.

Proof: To begin with, it is evident that we may use a section of the real axis of the w-plane instead of the straight line g; or else we should first perform an appropriate rotation and shift of the w-plane. Now, $\bar{w}=\tilde{w}=u-iv$ is the conjugate value to $w=u+iv$, and the function to be considered is

$$w=\tilde{f}(\tilde{z}),$$

i.e., the conjugate function of the conjugate argument. But this function is analytic with $w=f(z)$, because the Cauchy-Riemann differential equations (11) remain unchanged if we interchange v and $-v$ and, simultaneously, y and $-y$. Let now $W=F(z)$ be that function which coincides with $w=f(z)$ in the upper half-plane and with $\tilde{w}=\tilde{f}(\tilde{z})$ in the lower half-plane (and along the real axis, of course, with w and $\tilde{w}$, both being identical at the axis). Then we have

$$\oint_{(C,C')}\frac{F(\zeta)d\zeta}{\zeta-z}=\oint_{ABCA}\frac{F(\zeta)d\zeta}{\zeta-z}+\oint_{AC'BA}\frac{F(\zeta)d\zeta}{\zeta-z},$$

for the distance AB is traversed twice in opposite directions; the correspondent integrals thus cancel out. But by Cauchy's integral formula (47)

$$\oint_{ABCA}\frac{F(\zeta)d\zeta}{\zeta-z}=\oint_{ABCA}\frac{f(\zeta)d\zeta}{\zeta-z}=2\pi i f(z)\quad\text{or}\quad=0,$$

depending upon z being situated on the upper or lower half-plane, and likewise

$$\oint_{AC'BA}\frac{F(\zeta)d\zeta}{\zeta-z}=\oint_{AC'BA}\frac{\tilde{f}(\zeta)d\zeta}{\zeta-z}=0\quad\text{or}\quad 2\pi i f(z).$$

Hence, by addition,

$$\oint_{(C,\,C')} \frac{F(\zeta)d\zeta}{\zeta - z} = 2\pi i F(z).$$

As has been shown in Sect. C, Art. 15, it follows from the above that $F(z)$ is an *analytic function* for any inside point of $(B+B')$; consequently, $\bar{f}(\bar{z})$ is the analytic continuation of $f(z)$. As a matter of fact, it is not at all necessary that $f(z)$ be analytic along the section AB of the real axis: the continuity is quite sufficient. All we have to do is to enclose AB in a parallel stripe of arbitrary narrowness and to use its boundaries when integrating. The resulting integrals differ arbitrarily little from the above integrals because of the continuity of $f(z)$.

The reflection principle gives rise to very many applications. Referring to the elliptic integral (182), which maps the upper half of the z-plane upon a rectangle of the w-plane, one may, for example, easily prove by double-reflection that z must be a doubly periodic function of w. This, however, shall be left to the reader.

PART II

APPLICATIONS

A. THE CONSTRUCTION OF ELECTRIC AND MAGNETIC FIELDS BY MEANS OF SOURCE-LINE POTENTIALS

BY W. SCHOTTKY, BERLIN

1. The Problem.

The most familiar applications of the theory of functions refer to two-dimensional potential fields, for, from Part I, Sect. A, Art. 8 we realize that any complex analytic function represents, within its region of regularity, a solution of the equation of potential. Next to the homogeneous fields, the simplest are the rotational-symmetric ones. Therefore, the first chapter of applications deals with rotational symmetric fields of simplest form. It will, furthermore, be shown that with the help of these fields a large number of practically important field-forms may be synthetically constructed.

2. Definition and Properties of the Source-Line Potential.

The source-line potential is understood to mean a two-dimensional function of potential, created by a uniformly charged infinitely long source-line. Let the trace of this source-line with the xy-plane be the point $P_0(x_0y_0)$, which thus is the only point of the xy-plane showing a singularity (Fig. 35). The source sends out a field, which is symmetrical to all sides. The function complying with this condition is the function $u=\ln r$, where $r=\sqrt{(x-x_0)^2+(y-y_0)^2}$ designates the distance of the variable point $P(xy)$ from the source-line. This function satisfies the condition $\Delta u=0$ in the total space, except at the point P_0. The proof for the above statement may be carried out by direct calculation; from the standpoint of function theory the proof is superfluous, because $\ln r$ represents the real part of the function $\ln(z-z_0)$. Therefore, by Cauchy-Riemann's equations, $u=\mathrm{Re}\ln(z-z_0)$* satisfies Laplace's equation

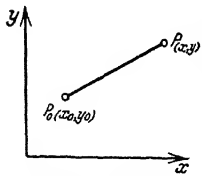


Fig. 35. Source-line and variable point.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

* Here, as in the following, Re is the real part while Im is the imaginary part of a complex function.

outside of $z=z_0$. As a matter of fact, not only $\ln(z-z_0)$ is a solution of the equation of potential, but also $K \ln(z-z_0)$. The introduced constant K possesses a simple physical meaning. To illustrate, let us consider along with the potential the field vector

$$\mathrm{A}=-\operatorname{grad} u. \tag{2}$$

The "divergence" of this vector shall be understood to be the integral

$$\oint \mathrm{A}_n ds, \tag{3}$$

where A_n denotes the field component at right angles to the line element ds (Fig. 36). If we calculate the divergence for a circle of radius r_0, we set for the field vector

Fig. 36. Divergence of a vector.

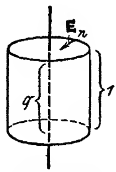
$$\mathrm{A}=-\frac{Kd\ln r}{dr} \cdot \frac{K}{r_0}, \tag{4}$$

and hence have

$$\oint \mathrm{A}_n ds=-\frac{K}{r_0}\oint ds=-2\pi K. \tag{5}$$

3. Interpretation of the Source-Line Potential as Electric Potential φ.

Let us now think of u representing an electric potential φ and let us define the electric field strength in the usual manner:



$$\mathbf{E}=-\operatorname{grad} u=\mathrm{A}. \tag{6}$$

We also calculate the integral

$$\oint \mathrm{E}_n ds=-2\pi K. \tag{7}$$

Fig. 37. Charge of a source-line.

We may visualize this integral geometrically (Fig. 37) as an integral of the lateral area of a circular cylinder of height 1. The charge q inside of an element of space of height 1 is given by

$$q=\Delta\oint \mathrm{E}_n ds. \tag{8}$$

If we restrict our consideration to the empty space, Δ represents the constant $\frac{1}{4\cdot\pi\cdot 9\cdot 10^{11}}$. With respect to equation (7), we obtain

$$q=-2\pi\Delta K. \tag{9}$$

Using these constants, the electric source-line potential becomes

$$\varphi = -\frac{q}{2\pi\Delta}\ln r. \tag{10}$$

The electric field strength in a radial direction has the value

$$-\frac{\partial\varphi}{\partial r} = \frac{q}{2\pi\Delta}\cdot\frac{1}{r}; \tag{11}$$

in a circular direction its value is zero. The potential surfaces thus are circular cylinders about the source-line through P_0, the lines of force are the radial rays $\vartheta = \text{const}$ at right angles to the cylinder surface. The complex function, from which these two families of lines may be derived in the complex xy-plane, is the function

$$f = K\ln(z - z_0), \tag{12}$$

to which belong the two component functions:

$$\varphi = K\ln r \tag{13a}$$

and

$$\psi = K\vartheta. \tag{13b}$$

4. The Construction of Electric Fields by Means of Source-Line Potentials. The Cylindrical Condenser.

Neither infinitely extended nor infinitely narrow electric conductors exist. Nevertheless, the computation of the field of such an idealized conductor may be utilized for applications. The condition of infinite extension for a mean region of a conductor may be substituted by the condition that the boundary effects are no more perceptible in the investigated region. This question should, of course, be tested (from case to case) by means of three dimensional approximation-considerations. Concerning the idealization of the infinitely narrow conductor, however, we may in many cases rid ourselves of it in a simple manner and go over to problems with finite conductor dimensions. Consider, for example, once more a single source-line with the function of potential $\varphi = K\ln r$ and the function of line of force $\psi = K\vartheta$. In the x, y-plane we then obtain a field configuration consisting of circles and radii and having a singularity in the center (Fig. 38). Let us now consider not the total field

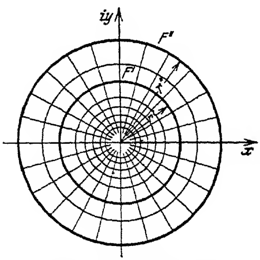


Fig. 38. Cylindrical condenser.

of force, but only the space between two potential surfaces F' and F'' with the radii r' and r''. Between these areas the equation $\Delta\varphi=0$ is complied with, the potential on both surfaces is constant, between them exists a potential difference $\varphi''-\varphi'$ to which we may assign any arbitrary value by appropriate choice of K. But these are exactly the conditions from which to determine the field between two concentric and cylindric conductors with the radii r_1 and r_2; here, too, the potential on each cylinder surface is constant, between the surfaces the equation $\Delta\varphi=0$ holds and a definite potential difference is given. As, moreover, we may always show in such cases that a solution, satisfying the given boundary conditions, is at the same time the only and true solution, the field of a source-line in the center of both cylinders at the same time represents the true field in the space between the cylinders.

With appropriate choice of K we thus have:

$$\left.\begin{aligned}\varphi' &= -K\ln r'\\ \varphi'' &= -K\ln r''\end{aligned}\right\} \tag{14}$$

and, therefore, the *voltage*

$$U=\varphi''-\varphi'=-K\ln\frac{r''}{r'}. \tag{14a}$$

From this we can calculate

$$K=\frac{-U}{\ln\dfrac{r''}{r'}}. \tag{15}$$

Hence,

$$\varphi=\frac{U}{\ln(r''/r')}\cdot\ln r. \tag{16}$$

The relation between K and U has, however, not only a significance for our calculations, but K is essentially proportional to the charge q (compare equation (9)). Thus, q is at the same time the charge pertaining to the potential difference U of the cylindrical condenser in question and

$$\frac{q}{U}=\frac{2\pi\Delta}{\ln\dfrac{r''}{r'}} \tag{17}$$

is, by definition, the capacitance per unit length of the above cylindrical condenser (as far as the edge effects may be neglected). If the medium between the cylinders possesses the dielectric constant ϵ, the capacity, as is well known, increases ϵ times.

5. Combination of Fields Due to Several Source-Line Potentials.

Of equal interest are fields, resulting from superposition of several source-line potentials. To begin with, these potentials again represent fields of infinitely narrow straight parallel lines of charge with the charges q_1, q_2, etc. For, combining the potentials φ_1, φ_2, etc., to a potential φ, the condition $\Delta\varphi=0$ is again complied with outside of the lines of charge, while the divergence $\int \mathbf{E}_n ds$ of the field around each line possesses the value determined by the corresponding charge. As in the case of the cylindrical condenser, however, we may restrict our consideration to the space between certain potential surfaces or potential lines, thus gaining new forms of conductors and, simultaneously, the solution of the special potential problem.

A simple example of such nature utilizes the combinations of two source-lines oppositely charged and going through the points P_1 and P_2 (Fig. 39). The field of these source-lines for a variable point P may be calculated in the real plane by adding-up the functions φ_1 and φ_2:

$$\varphi=\varphi_1+\varphi_2=-\frac{q}{2\pi\Delta}\cdot\ln r_1+\frac{q}{2\pi\Delta}\cdot\ln r_2=\frac{q}{2\pi\Delta}\cdot\ln\frac{r_2}{r_1}. \qquad (18)$$

From this, the following construction of equipotential lines and of lines of force results: we have to draw-in the lines of potential, along which the ratio of distances from the points P_1 and P_2 is constant, and then the lines of force at right angles to the lines of potential. We thus obtain a configuration, consisting of circular cylinders intersecting at right angles. We realize that, in analogy with the case of a cylindrical condenser, we may again interpret two of the appearing circles as conductor surfaces, in this manner obtaining the distribution of potential between oppositely charged circular cylindrical conductors, which, however, now may assume an arbitrary position and dimension relatively to each other. Thereby, the problem of mutual capacitance of two parallel circular cylinders may also be rigorously proved. These calculations are, for example, applied to the computation of the capacitance of a two-conductor overhead line between one conductor and ground (limiting case of an infinitely large radius of curvature of one of the surfaces), and in similar cases.

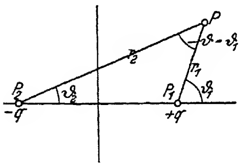


Fig. 39. Referring to the field of two source-lines.

The method of function theory deepens the treatment of such an

example, because, when using this method, we do not add the functions of potential but, appropriately, the functions w_1 and w_2 of the functions of potential and of line of force. Hence,

$$w=\varphi+i\psi=w_1+w_2=K\cdot\ln(z-z_1)-K\cdot\ln(z-z_2). \tag{19}$$

The real part of this function again satisfies equation $\Delta u=0$ and has an opposite equal divergence at the source-line points P_1 and P_2; this does not furnish anything new, as Re ln $(z-z_1)=\ln r$, and we thus are led to exactly the same representation as when using the logarithmic potential in the real plane. New, however, is the use of the function of line of force $\psi=\psi_1+\psi_2$. As we generally know this function, by means of the lines $\psi=$const, furnishes the system of lines of force at once in a closed form. Now, $\ln(z-z_1)=\ln r_1+i\vartheta$, where ϑ is the angle between the vector of the variable point and a fixed axis, most conveniently the axis going through P_1 and P_2, which we choose as the x-axis (Fig. 39). Therefore, $\psi=\vartheta_1-\vartheta_2$ is simply the difference of the angles between the rays r_1 and r_2 and the fixed axis, thus the angle between r_1 and r_2. The lines of force, hence, represent curves, along which the section $P_1\,P_2$ appears under an always equal angle; these curves are, as is well known, the totality of circular arcs over $P_1\,P_2$ (Fig. 40).

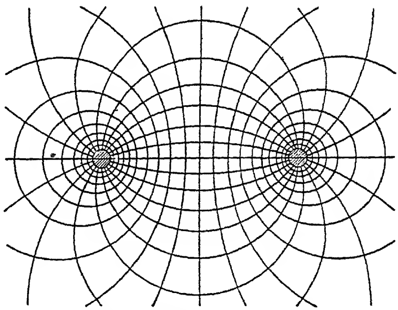


Fig. 40. Field plot due to two source-lines.

The theory of functions, however, points (more directly than the treatment in the real plane) towards still another way of exploitation of a given field of forces for its applications. The function $-iw=\psi-i\varphi$ may be treated in the same manner as the function w; the lines $\psi=$const may be interpreted as lines of potential, the lines $\varphi=$const— as lines of force. Referring to our special problem, we would thus find the field between two circular arcs over the same chord, the arcs being kept at different potentials.

6. The Grid Potential of an Amplifier Tube.

As a further application, we shall now construct the field in an amplifier tube by means of source-line potentials. Let us consider a

3-electrode tube with cylindrical arrangement as indicated in Fig. 41. Let the electrodes be long relative to their diameter, and let the grid consist of n wires parallel to the axis (radius ρ_g). The cathode (radius ρ_k) is located in the axis of the cylinder. Let the charge per unit length of the cathode be $+q_k$, that of the grid wires being each equal $-q_g$. For the time being we calculate only the potential of the cathode charge and of the grid charges, neglecting the space charge of the electrons. We assume the charge to be concentrated in the axis of the respective wires. Then, the potential of each individual wire is a source-line potential of the discussed nature; we should, however, note the dislocation of the source-line away from the origin of the system of coördinates. Let $z_{gl}=z_{lg}$ designate the position vector of the source-line χ of the grid. The source-line thus has the complex potential

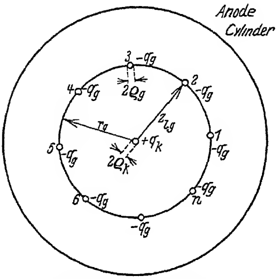


Fig. 41. Arrangement of electrodes in a 3-electrode tube.

$$\chi_l = + \frac{q_g}{2\pi\Delta} \ln (z - z_{gl}). \tag{20}$$

For the totality of all grid wires we get

$$\begin{aligned} \chi_g = \sum_1^n \chi_l = + \frac{q_g}{2\pi\Delta} \sum_1^n \ln (z - z_{gl}) \\ = + \frac{q_g}{2\pi\Delta} \ln \left((z-z_{g1})\ (z-z_{g2}) \cdots (z-z_{gn})\right). \end{aligned} \tag{21}$$

The product under the logarithm sign may be computed in a closed form; for, as the z_{gl} values form the corners of a regular polygon inscribed in the circle of radius r_g, we have

$$(z-z_{g1})\ (z-z_{g2}) \cdots (z-z_{gn}) = z^n - z_g^{\ n} \tag{22}$$

Hence,

$$\chi_g = + \frac{q_g}{2\pi\Delta} \ln (z^n - z_g^{\ n}) = + \frac{nq_g}{2\pi\Delta} \ln \sqrt[n]{z^n - z_g^n}\,. \tag{23}$$

The potential of the cathode charge is superimposed on the above:

$$\chi_k = \frac{-q_k}{2\pi\Delta} \ln z. \tag{24}$$

The total potential thus becomes:

$$\chi=\chi_k+\chi_g=-\frac{q_k}{2\pi\Delta}\ln z+\frac{nq_g}{2\pi\Delta}\ln\sqrt[n]{z^n-z_g^n}. \tag{25}$$

As for the field pertaining to this potential we may state the following. In the immediate vicinity of the grid points 1, 2 etc., provided that q_g and q_k are not of too different a magnitude, the influence of the corresponding source-line is always predominant. And this the more so, as in the neighborhood of the cathode the rest of the fields compensate each other to a large extent and in the neighborhood of the grid points they do so at least quite appreciably. If we, therefore, describe circles about the centers of the individual seats of source with radii, which are small as compared to the distance to the nearest source-point, $\ln|z|$ and $\ln|z-z_{gl}|$ respectively are approximately constant on these circles and, this being the by far predominant term of the potential, also $\varphi=\text{const}$ on these circles. Furthermore, the potential represented by equation (25) furnishes, sufficiently far away from all source-lines, approximately circular cylinders as equipotential surfaces. It, therefore, is permissible to let one of these cylinders coincide with the anode cylinder to which we assign the anode potential φ_a. By appropriate choice of q_k and nq_g, we may assign any arbitrary value to the potential differences $\varphi_g-\varphi_k$ (grid voltage) and $\varphi_a-\varphi_k$ (anode voltage); or, inversely, we also may determine the charges q_k and nq_g pertaining to the above values of potential.

7. Calculation of the Amplification Factor of an Amplifier Tube.

It is true that the relation between charges and potential difference in an amplifier tube interests us in a certain way, particularly enabling us to calculate the component-capacitances of the three electrode system. By far the most important value, however, is the so-called "*Durchgriff*."* The calculation of the "Durchgriff" shall now be carried through, utilizing a method possessing the advantage of enabling us to demonstrate the transition to fields including space charge in a particularly clear manner.† Again we assign the potentials φ_k and φ_a to cathode

* Translator's remark: This German word, in literal translation meaning the "grasping through," has no parallel in English but is exclusively used in the German literature. It has, therefore, been retained in the translation. The English literature on vacuum tube theory uses the so-called "amplification factor" instead. The relation between both values is the following:

$$D=\text{Durchgriff}=\frac{1}{\text{Amplification factor}}=\frac{1}{\mu}.$$

†W. Schottky, "Über Hochvakuumverstärker," *Arch. Elektrot.* Vol. 8 (1919), pp. 1, 299. From studies carried out in summer 1915.

and anode, but temporarily assume the grid wires to be replaced by a homogeneous closed grid-cylinder G at piloting potential φ_{pil}. This potential shall be chosen in such a way that the field in the neighborhood of cathode and anode attains that value, which it actually would attain due to the grid at potential φ_g. The above is always possible.

We first calculate the total charge, i.e., the sum of inside and outside charge carried by the grid, $\varphi_{pil}-\varphi_k$ and $\varphi_a-\varphi_{pil}$ being given. The outside charge per unit length is

$$q' = -\Delta \oint \mathrm{E}_g' ds, \tag{26}$$

where E_g' is the field strength of the anode-grid space pointing towards the grid cylinder. This field strength being constant along the grid cylinder, equation (26) yields

$$q' = -\Delta \mathrm{E}_g' \oint ds = -\Delta \mathrm{E}_g' 2\pi r_g. \tag{27}$$

Correspondingly, the inside charge per unit length is

$$q'' = +\Delta \oint \mathrm{E}_g'' ds, \tag{28}$$

where E_g'' is the field strength of the grid-cathode space pointing towards the cathode. Now, the anode-grid space and the grid-cathode space are electrostatically shielded from each other by the homogeneous grid cylinder; the cathode charge $+q_k$, therefore, is determined by E_g'' alone and is equal to $+q''$. In equation (25), thus, the first item on the right-hand side is equal to the complex piloting potential χ_{pil}, and we may write:

$$\chi - \chi_{pil} = + \frac{nq_g}{2\pi\Delta} \ln \sqrt[n]{z^n - z_g^n}\,. \tag{29}$$

From the above follows the grid potential φ_g itself, by putting $z = r_g + \rho_g$ ($\rho_g << r_g$) and splitting-up the real component:

$$\begin{aligned} \varphi_g - \varphi_{pil} &= + \frac{nq_g}{2\pi\Delta} \mathrm{Re} \ln \sqrt[n]{(r_g+\rho_g)^n - r_g^n} \\ &= + \frac{nq_g}{2\pi\Delta} \ln \left| \sqrt[n]{r_g^n + nr_g^{n-1}\rho_g + \cdots - r_g^n} \right| \\ &= \sim + \frac{nq_g}{2\pi\Delta} \ln \left(r_g \sqrt[n]{\frac{n\rho_g}{r_g}} \right) \end{aligned} \tag{30}$$

q_g being composed of q' and q''. But the component charge q' may be computed from the potential difference $\varphi_a - \varphi_{pil}$ of the anode-grid space by means of the cylindrical condenser formula, while q'' is given by

equation (28). Hence, r_g designating the grid radius and r_a the anode radius, we have:

$$\varphi_g - \varphi_{pil} = (\varphi_a - \varphi_{pil}) \frac{1}{\ln \frac{r_a}{r_g}} \ln \left(r_g \sqrt[n]{\frac{n\rho_g}{r_g}} \right) - \frac{n}{\Delta} r_g \mathrm{E}_g'' \ln \left(r_g \sqrt[n]{\frac{n\rho_g}{r_g}} \right). \tag{31}$$

Assuming that E_g'' depends *linearly* upon φ_{pil}, we may follow from equation (31) that

$$\text{const} \cdot \varphi_{pil} = \varphi_g + \frac{\ln \frac{nr_g}{\rho_g}}{n \ln \frac{r_a}{r_g}} \cdot \varphi_a. \tag{32}$$

Differentiation thus furnishes:

$$\frac{\partial \varphi_{pil}}{\partial \varphi_a} : \frac{\partial \varphi_{pil}}{\partial \varphi_g} = \frac{\ln \frac{nr_g}{\rho_g}}{n \ln \frac{r_a}{r_g}} = D. \tag{33}$$

But this is exactly the ratio appearing in the general amplifier theory as the "Durchgriff" factor D; it determines the ratio of the current change due to grid potential to the current change due to anode potential, the current only depending upon the piloting potential φ_{pil} (by way of the equation of space charge). The above example demonstrates in a general form the calculation of the "Durchgriff" for cylindrical grids with longitudinal wires.

The reasons why we did not compute E_g'' is, first, because it cancels out when calculating D, and second, because the calculation of E_g'' from the cathode field free of space charges certainly appears to be inadmissible. The author has shown* that a relation between E_g'' and φ_{pil} must rather be computed from the theory of space-charge, leading to certain statements about the "piloting rigour" of a grid. If a grid is surrounded from both sides by extended electrodes and spaces free of space-charges, then we may, of course, calculate E_g'' analogously to E_g'.

8. Computation of the Field of an Amplifier Tube by Means of the Conformal Mapping $z = \sqrt[n]{Z}$.

Till now we have constructed the grid potential out of source-lines. The same formula may also be ascertained by means of a function-theoretical procedure.† We start out with a function w in the

* W. Schottky, "Zur Raumladungstheorie der Verstärkerröhren," *Wiss. Veröffentl. a. d. Siemens Konzern*, Vol. 1, Heft. 1 (1920), p. 64.

† M. Abraham, "Berechnung des Durchgriffes von Verstärkerröhren," *Arch. Elektrot.*, Vol. 8 (1919), p. 42.

$Z=(X+iY)$-plane (Fig. 42). Instead of considering the function w of Z, we investigate the function $\sqrt[n]{Z}=z=x+iy$. If we plot the z-values in a new complex plane xy, n points in the xy-plane correspond to *one* point in the XY-plane. This may be realized by following reasoning: the total Z-plane is mapped upon an angular region $\frac{2\pi}{n}$, so that the complete z-plane contains n exemplars of the Z-plane side by side or, which amounts to the same, the complete Z-plane contains one exemplar of the total z-plane in n sheets one above the other.

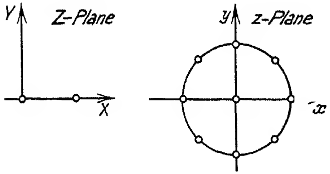


Fig. 42. Construction of the grid field by means of conformal mapping.

We easily recognize that we may arrive at a function of potential of a circular grid with central source-line, if we consider (in the Z-plane) a function of potential due to a source at the origin and a source with different charge at a certain distance from the origin (for instance at the X-axis). Then, the last mentioned source will appear n times on the periphery of a circle when the transformation is being carried through; the mean value of charge will always appear at the same place, however, with n-times its strength. The distances of the nucleus source-lines and the grid source-lines will be distorted in a definite way, but then again, we may suitably choose the original distance. The importance of the result is that we automatically obtain for the potential of the grid source-line a closed representation, while, when summing up the individual potentials, the above potential was obtained in the unclear form of a logarithmic sum of different distances. The grid point 1 in the Z-plane, namely, is characterized by the term $\ln(Z-Z_1)$ of the w-function; transferring this into the z-plane, we at once obtain, without use of the equation of circle-division, $\ln(z^n-z_1^n)$. The real part of this function is (but for the charging-constant) the potential of the circular grid. Of particular interest is the value of this potential function on the circles (of radii ρ_g) about the grid points; this value has been calculated in equation (30).

9. Potentials of Plane Series of Source-Lines.

Along with the cylindrical grid arrangements, the plane grid arrangements, consisting of a great number of parallel wires, play an important part in the various fields of applications; these plane grids may be represented by an infinite series of equidistant, equally or oppositely charged source-lines. To compute the potential of such an infinite

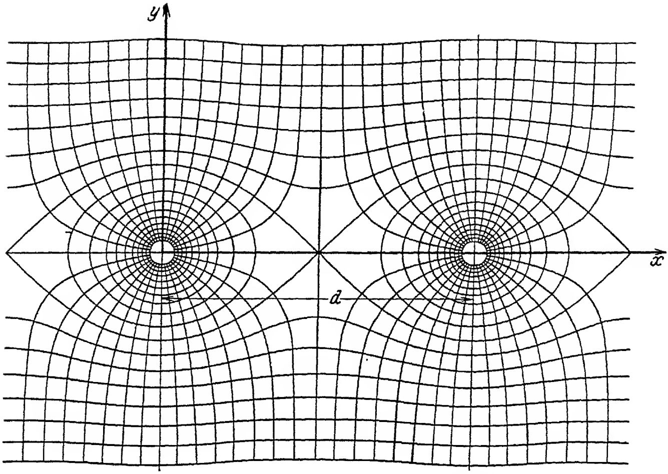


Fig. 43. Plot of the potential of an infinite series of equally charged source-lines.

series by means of ordinary addition of the individual logarithmic potentials — is a cumbersome procedure; the theory of functions, however, enables us to simplify the calculations considerably.

Consider the function $w = \ln \sin z$, a function being regular throughout the complete z-plane, except at the points where $\sin z$ happens to be equal zero. At these points, however, that is for $x=0$, $x=\pi$, $x=-\pi$, etc., the function behaves exactly as the function $\ln(z-z_0)$ and, thus, as an ordinary logarithmic source-line potential. This is readily conceived for $x=0$. For $x=\pi$, etc., the real part of the function repeats itself so that the same conditions as at $x=0$ exist at that point and at the other zeros. The real part of w, therefore, represents a potential function, satisfying the equation $\Delta\varphi=0$ at all points of the x-axis, except at the points 0, π, etc.; at these points the potential function behaves, as if source-lines of always equal charges would exist at them. If we form $w = -\frac{q}{2\pi\Delta} \cdot \ln \sin z$, then q again designates the charge per unit length of the source-lines. The solution of the potential being a one-valued one, Re w actually represents the potential function, indicating in a closed form the potential of the equidistant source-lines distributed along the x-axis; writing $\frac{\pi z}{d}$ instead of z, the source-lines are located at the points $\frac{\pi x}{d} = k\pi$, i.e., $x=d$, $2d$, etc. (Fig. 43).

Again making use of the property that the potentials are nearly constant within small circular cylinders about the source-points and

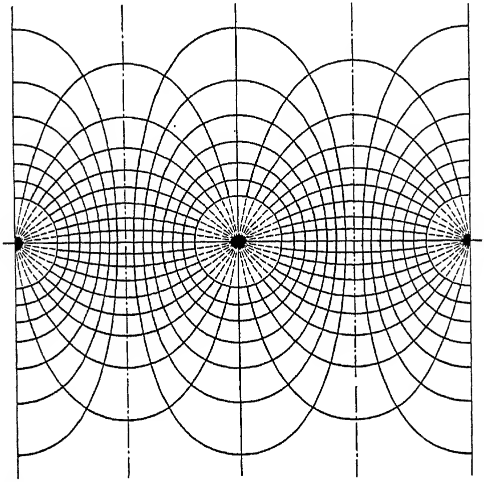


Fig. 44. Field plot of a series of alternatively positive and negative source-lines.

using the symmetry of the function with respect to each grid point, we obtain (by means of the above-mentioned function) the solution of the potential problem concerning an infinite series of circular cylinders of constant potential.

We may, of course, just as well start from the function $\ln \cos z$. This does not furnish a new field plot. The combination

$$\ln \sin z - \ln \cos z = \ln \frac{\sin z}{\cos z} = \ln \tan z,$$

on the contrary, furnishes in its real part the arrangement of alternatively positive and negative source-lines (compare Fig. 44).

10. Applications of Infinite Source-Line Series.

As an application of the potential of plane series of source-lines, let us consider the current and voltage distribution in a house-wall,* in the central axis of which is located the entrance spot of a current (for instance due to the break-down of an insulator). This problem is not

* F. Ollendorf, "Elektrische Stromleitung an feuchten Gebäudewänden," *Arch. Elektrot*, Vol. 19 (1927), p. 124.

an electrostatic one, but the equations and the boundary conditions are exactly the same; the electric potential satisfies the equation $\Delta\varphi=0$ outside of the source-point, and analogous relations exist between the value of the current, the potentials and field intensities as between charges and potentials in a dielectric. As favorable circumstance for our consideration we note that, in first approximation, we may altogether neglect the depth extension of this potential field, so that we are dealing here with an actually two-dimensional problem. If we assume the house-wall to be infinitely high, the distribution of potential in the wall is reproduced by a section of the total picture of the field of a uniformly charged series of points, this section screening-off of the picture a source-point with its corresponding infinitely long stripe (Fig. 45, upper half); for, no lines of force go here through the boundaries of this stripe from one source-point to another — by reasons of symmetry; analogously, it may be assumed in first approximation that no lines of force pierce through the boundaries of the house-wall. The picture of lines of force corresponding to Fig. 43 is, however, not the final one; actually, the current does not flow to the top and to the bottom, but all currents are directed towards ground. We may achieve this in a simple manner, reflecting our potential function at the surface (Fig. 45, lower half) and then considering only the field above the ground. To do this, we have to use the function $w=\text{const}\cdot[\ln\sin z-\ln\sin(z-z_0)]$, where the original source-point is assumed to be $z=0$ and z_0 represents the center with the coördinates 0 and $-2h$ (h=height of the source-point above ground). The real part of this function furnishes the final potential function as indicated in Fig. 46.

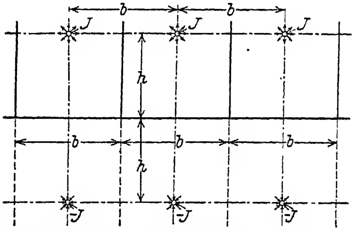


Fig. 45. Referring to the calculation of currents in a house-wall.

Another application, also of non-electrostatic nature, may be found in the literature for the case of alternating series of source-lines. With the help of this field plot we may idealize the leakage field between relatively long adjacent pole tips of a multi-pole machine. The computation has been carried out by Hague.* The field indicated in Fig. 44 shows the course of the potential and of the lines of force as a clipping from the infinite alternating series of source-lines.

* *J. Inst. El. Eng.*, Vol. 61 (1923), p. 1072; compare also F. Noether, "Über eine Aufgabe der Kapazitätsberechnung," *Wiss. Veröffentl. a. d. Siemens Konzern*, Vol. 2 (1922), p. 198.

From electric applications of potential functions of an extended series of source-lines between parallel massive electrodes, the computation of the field in electric precipitators should be mentioned. Here, we have a series of thin wires or net-like electrodes, acting as negative charged "scintillating electrodes" between metallic walls. If the currents are weak, that is as long as there are no appreciable space

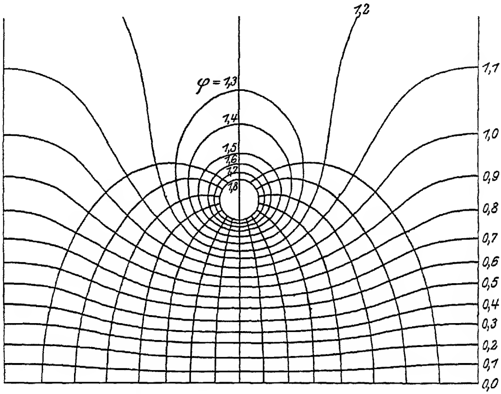


Fig. 46. The path of currents in a house-wall.

charges, the appearance of the minimum field strength necessary for corona formation and, with it, for the starting of self-discharge, may be calculated from the potential theory; in most cases, to be sure, we will be permitted to neglect the influence of adjacent charges and to deal with the simple case of a wire between two electrodes, or a wire within a cylinder. If, on the contrary, the currents grow stronger (being of the magnitude of 1 mA per meter scintillating wire), the electrostatic calculations lose their validity altogether.

11. The Part of Source-Line Potentials in the Calculation of Fields Due to Currents.

We now turn to the calculation of magnetic fields of infinitely extended linear current paths. They either may extend in air or, and that is by far the more important case, inside or outside of iron parts to which we assign an equal form at all cross-sections and an infinite extension in a direction normal to the $x\,y$-plane. Problems of this nature furnish important approximation calculations of fields of transformer windings having rectangular iron cross-sections, as well as of fields of

motors and generators, the axial dimensions of which are large relative to their radial dimensions.

As is well known, the field of a linear conductor may be derived from a vector potential V. The vector potential for the x-direction, for example, is related to the current density of the conductor in the z-direction* in the same manner as the electrostatic potential is related to the charge density in an electrostatic field. Thus, for an infinitely long wire, carrying the current I, the vector potential has only a direct component

$$V_z = -2I \ln r.$$

while the normal components V_x and V_y vanish:

$$V_x = 0, \; V_y = 0.$$

For the magnetic field strengths in the x- and y-direction the relations hold:

$$H_x = \mathrm{rot}_x\, V = +\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} = +\frac{\partial V_z}{\partial y},$$

$$H_y = \mathrm{rot}_y\, V = +\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} = -\frac{\partial V_z}{\partial x}.$$

The potential of a source-line satisfies the equation $\Delta\varphi = 0$ in the whole region outside of the source; thus, in the present case, $\Delta V_z = 0$. If we now form a complex function w, containing the potential function V_z as its real part, the corresponding imaginary part ψ of w is again a potential function, which, by virtue of the Cauchy-Riemann equations

$$\frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x}; \qquad \frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y}$$

satisfies the conditions:

$$\frac{\partial \psi}{\partial y} = +\frac{\partial V_z}{\partial x} = -H_y,$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial V_z}{\partial y} = -H_x.$$

Hence, $H = -\mathrm{grad.}\,\psi$. For two-dimensional problems it therefore seems that, as far as electro-magnetism is concerned, the representation by means of function theory is perhaps of still more comprehensive significance than that dealing with electrostatic problems; both parts of a complex function have a direct physical meaning in those parts of the space in which V_z satisfies the equation $\Delta V_z = 0$: the real part represents the vector potential, the imaginary part — the scalar potential. The

* Here and in the following formulas the third coördinate of a cartesian system of coördinates x, y, z in space.

magnetic field strength may be derived from the scalar potential in the same manner as the electric field strength from the electric potential. There exists, to be sure, the fundamental difference that the part $\psi=\text{const}\ \vartheta$ of the logarithm function is not one-valued, but rather increases by 2π with each completed circuit about the source-line. This is an important property of any scalar stream-line potential. The vector potential, on the contrary, being the function assigned to the potential ψ, determines the course of the lines of force: $V_z=\text{const}$ is the equation of the system of lines of force.

Combining source-line vector potentials, we may determine the joint magnetic fields of many arbitrary straight conductors in air; thus, for example, the magnetic field of a double overhead line, the magnetic field of a multi-wire antenna, the magnetic field of a series of cylindrically arranged wires or of a series of equally or oppositely directed currents. The same functions as used in the electrostatic case may, of course, be used here. In such cases, the ever equal coördination of electric and magnetic fields, is the reason why the same relation always exists between the inductivity and capacity of such straight conductors; from this, the ever equal velocity of propagation of electro-magnetic waves along straight wires in air may be derived.

If we wish to compute the obtained magnetic fields for a part of the space only (similar to the case of the cylindrical condenser), we have, to be sure, to pay attention to the fact that the surface of complete magnetic conductors of permeability $\mu=\infty$ must coincide with the surface $\psi=\text{const}$ and not with $\varphi=\text{const}$. Starting out with the electrostatic field plot, we therefore must identify the surface of the complete magnetic conductor with an electrostatic line of force. From this it follows, for example, that the image of a conductor parallel to a plane iron surface, is given by introduction of a *positive* and not of a negative image.

12. Calculation of Current-Fields between Axially-Symmetric Iron Slabs.

The method of images is practically most important for ferromagnetic bodies, for which $\mu=\infty$ may be taken in first approximation. Another interesting method of function theory shall, therefore, be outlined, a method meeting far-reaching requirements in the calculation of fields of machines with circular cylindrical rotors and stators. This problem may be reduced to the calculation of the field of a single wire situated at some point of the rotor or between rotor and stator as indicated in Fig. 47; to accomplish this we make use of a process of superposition.

Problems of such nature have especially been treated by Searle* and Hague.† The total magnetic field is composed of two parts; the primary field is due to the wire itself, the secondary field is given by the magnetization of the iron due to the field of the wire and may be derived from magnetic "surface-layers" at the boundaries of the iron body if we assume $\mu=$const. Consequently, this part of the magnetic field (within any zone of equal magnetic properties) may be derived from an ordinary potential, satisfying the equation $\Delta\varphi=0$ and, thus, being interpreted as the real part of a complex function w_m. Accordingly, we shall also have to choose a complex function for the field of the wire, the real part of which directly furnishes the potential ψ; the primary potential is $-iw=\psi=iV_z$, which is proportional to $-i\cdot\ln z$ (for a single wire).

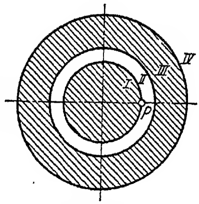


Fig. 47. Referring to the calculation of the magnetic field of an electric machine.

The problem now reduces to the determination of the function of the secondary potential w_s for each of the four zones *I*, *II*, *III*, *IV* (Fig. 47). In each zone this function evidently must be symmetrical with respect to the line going through the wire and the axis of the figure, i.e. to the line OP. It may be shown that, in the most general case, such a function may be composed by the action of multi-poles of different order arranged in the central axis. For, if u is the potential function of a pole, $\frac{\partial u}{\partial x}$ determines the potential function of a dipole located in the x-direction, $\frac{\partial^2 u}{\partial x^2}$ that of a quadrupole, etc. If we differentiate the function w of a pole with respect to the complex variable z, we have $\frac{\partial w}{\partial z}=\frac{\partial u}{\partial z}+i\frac{\partial v}{\partial z}$ and, therefore, respecting the analytic character of w, $\frac{\partial w}{\partial z}$ is also equal $\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$. As this representation simultaneously separates the real from the imaginary, the above indicates, that $\mathrm{Re}\,\frac{\partial w}{\partial z}$ determines the potential of a dipole in the x-direction; corresponding relations hold for the dipoles of higher order. Carrying out the calculation for the dipole, we get:

* *Electrician*, Vol. 11 (1928), pp. 453, 510.

† "Electromagnetic Problems in Electrical Engineering," London: Oxford University Press, 1929.

$$\frac{\partial w}{\partial z} = \frac{1}{z} = \frac{1}{r} \cdot e^{-i\vartheta} = \frac{\cos \vartheta}{r} - \frac{i \sin}{r}$$

$$\text{Re } \frac{\partial w}{\partial z} = \frac{\cos \vartheta}{r}.$$

In general we have:

$$\frac{\partial^n w}{\partial z^n} = \text{const } \frac{1}{z^n} = \text{const} \left(\frac{1}{r^n} \cdot e^{-in\vartheta} \right) = \left(\frac{\cos n\vartheta}{r^n} - \frac{i \sin n\vartheta}{r^n} \right) \cdot \text{const}$$

and, therefore,

$$\text{Re } \frac{\partial^n w}{\partial z^n} = \text{const } \frac{\cos n\vartheta}{r^n}.$$

The function $\cos n\vartheta$ actually has the property to be symmetrical to $\vartheta = 0$. The complete representation of the magnetic field is given by the function:

$$w_m = A_0 + A_1 \cdot \ln z + A_2 \cdot \frac{1}{z} + A_3 \cdot \frac{1}{z^2} + A_4 \cdot \frac{1}{z^3} \text{etc.},$$

where the signs and the numerical constants are contained in the A_k-values. This representation, to be sure, is still a special one, inasmuch as all the multi-poles are located at the zero point; poles at any other point would violate the symmetry, but multi-poles at infinity do not violate it. We thus may add to w_m poles issued from the given pole by the transformation $z' = \frac{1}{z}$; this does not furnish anything new for the first two terms, but for the following terms — the new forms $z, z^2 \ldots z^n$ with the real parts $r^n \cdot \cos n\Theta$. As a whole, we have for the real part a Fourier decomposition of the function (periodic in ϑ) with components containing the terms $\frac{1}{r^n}$ and r^n; in complex representation:

$$w_s = A_0 + A_1 \cdot \ln z + A_2 z + \frac{B_2}{z} + A_3 z^2 + \frac{B_3}{z^2} + \cdots$$

The real part of this function furnishes us the most general symmetric potential functions; making the coefficients of the 4 zones different, it is possible to express the field at both sides of the boundary line (in a normal and tangential direction) by means of the coefficients A and B. To this we would have, of course, to add the field of the wire itself, which may be derived from the function w_m.

The boundary conditions require the continuity of the field components parallel to the boundary surfaces: moreover, the induction normal to the boundary surface must remain continuous. Thus, the self-explanatory relations $\mathbf{H}'_{\|} = \mathbf{H}''_{\|}$ and $\mathbf{H}'_{\perp} = \mu \cdot \mathbf{H}''_{\perp}$ exist. These conditions

have to be written down for all boundary surfaces and then furnish just enough equations to evaluate all of the coefficients A and B; thus, it is possible to express the constants and the position and strength of the current in terms of the geometric dimensions. By superposition of different potentials of the wires we then, of course, are enabled to determine the magnetic fields of complete windings.

13. The Limits of the Method of Function Theory.

Finally, let us briefly discuss the treatment of such two-dimensional problems in which the conductors no longer can be represented with sufficient accuracy by source-lines. These are, for example, problems concerning the magnetic field of conductors lying in rectangular slots of an armature; these are, further, the more exact problems concerning the fields and the leakage fields of transformers. Among the latter type of problems there is for instance the one of windings separated from each other in disk-like manner; these windings are considered as one single homogeneous current carrying conductor.

In all these cases a vector potential V_z, from which the magnetic field may be determined, is still existent, but this vector potential does not satisfy any longer the potential equation $\Delta V_z = 0$; inside the conductor, rather, the relation $\Delta V_z = -4\pi \mathbf{i}$ holds, where $\mathbf{i}$ designates the current density normal to the xy-plane. Thus, inside the conductor, V_z can not any longer be represented by the real part of a complex function, or, in other words, this function would have to behave singularly inside of the conductor cross-sections. It is evident that in such cases the use of complex functions may be retained only in quite an artificial way; here, the direct use of the potential function V_z is essentially simpler — at least inside of the zones in which the conductors are located.* In problems of such a nature the application of the methods of function theory is restricted to the calculation of the fields outside of the current carrying conductors.

B. TWO-DIMENSIONAL FIELDS OF FLOW

BY K. POHLHAUSEN, BERLIN

1. The Problem.

The following chapter deals with the application of function theory to the two-dimensional flows of ideal liquids. The point is to calculate the stationary field of flow around cylindrical bodies and to evaluate the

* W. Rogowski, *Mitteilungen über Forschungsarbeiten des V. D. I.*, Heft 71 (1909), pp. 1–36; *E. T. Z.*, Vol. 31 (1910), pp. 1033–1036, 1069–1071. E. Roth, *Bull. Soc. Franc. Electr.*, Vol. 7 (1927), pp. 840–966; *Rev. gén. électr.*, Vol. 22 (1927), pp. 417–424; Vol. 23 (1928), pp. 773–787.

forces acting upon those bodies. Of particular interest to aero-dynamics is the calculation of the lift exerted upon an (infinitely long) airfoil in a flowing medium. We shall take no account of the treatment of non-stationary processes and of three-dimensional flows.

This chapter requires but elementary knowledge of function theory: the intrinsic property of the complex function and its decomposition into real and imaginary parts. Aside of that, we shall confront a number of examples of application, supplementing and deepening our knowledge of conformal mappings.

It is assumed that the process of flow depends only upon two space coördinates. Furthermore, we neglect the friction of the liquid and consider it to be incompressible.

2. Fundamentals and Notations.

In the mathematical part of this book it has been shown that any complex analytic function $F(z)$ satisfies the partial differential equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=0 \tag{1}$$

and can be decomposed into a real and imaginary part, such that:

$$F(z)=\varphi(x,\, y)+i\psi(x,\, y). \tag{2}$$

Here, $\varphi(x, y)$ as well as $\psi(x, y)$ are real functions, themselves satisfying differential equation (1).

In applications dealing with the flow of fluids $\varphi(x, y)$ is called the *velocity potential* and $\psi(x, y)$ the *stream function.* Both are brought into relation by the *Cauchy-Riemann differential equations:*

$$\begin{aligned}\frac{\partial\varphi}{\partial x}&=\frac{\partial\psi}{\partial y},\\ \frac{\partial\varphi}{\partial y}&=-\frac{\partial\psi}{\partial x}.\end{aligned} \tag{3}$$

The curves $\varphi(x, y)=\text{const}$ are called *equipotential lines,* the curves $\psi(x, y)=\text{const}$ are called *stream lines.* By virtue of equations (3) both families of curves intersect at right angles.

The components of the velocity v of the irrotational flow given by $\varphi(x, y)$ are:*

$$\left.\begin{aligned}v_x&=\frac{\partial\varphi}{\partial x}=\frac{\partial\psi}{\partial y},\\ v_y&=\frac{\partial\varphi}{\partial y}=-\frac{\partial\psi}{\partial x}.\end{aligned}\right\} \tag{4}$$

* In fluid dynamics, thus, the field is represented as the positive gradient of potential, whereas in electrostatics the negative gradient is considered to represent the field.

Differentiating the complex function $F(z)$ with respect to z, we obtain, for example:

$$\frac{dF(z)}{dz}=\frac{\partial\varphi}{\partial x}+i\frac{\partial\psi}{\partial x}=v_x-iv_y \tag{5}$$

This complex quantity represents the physical velocity vector reflected at the positive x-axis and is called the *complex velocity*. For the absolute value of $\frac{dF(z)}{dz}$ we find:

$$\left|\frac{dF(z)}{dz}\right|=|F'(z)|=\sqrt{\left(\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2}=\sqrt{v_x^2+v_y^2}=|v|. \tag{6}$$

Thus the *absolute value of $F'(z)$ is equal to the absolute value of the velocity of flow.*

3. Axially-Symmetrical Flow.

Of the elementary functions, the function $F(z)=c\ln z$ is of importance as far as applications are concerned. This function represents two basically different fields of flow, depending upon c being real or imaginary.

(a) *The constant c is real; source-lines.* Putting $c=\frac{q}{2\pi}$ we obtain

$$F(z)=\frac{q}{2\pi}\ln z. \tag{7}$$

We call q the *discharge per unit length of source-line.* Now, the potential is the real part of $\frac{q}{2\pi}\ln z$, thus

$$\varphi=\frac{q}{2\pi}\ln r=\frac{q}{2\pi}\ln\sqrt{x^2+y^2}=\frac{q}{4\pi}\ln(x^2+y^2) \tag{8}$$

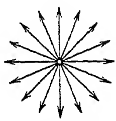


Fig. 48. Flow from a source-line.

and the stream function is the imaginary part, thus

$$\psi=\frac{q}{2\pi}\tan^{-1}\frac{y}{x}. \tag{9}$$

The equipotential lines are, therefore, circles about the piercing point of the source-line through the xy-plane, while the flow itself takes place radially to the outside or to the inside, depending upon q being positive or negative (Fig. 48).

(b) *The constant c is imaginary; vortex-lines.* Putting $c=-\frac{i\Gamma}{2\pi}$ we have

$$F(z)=-\frac{i\Gamma}{2\pi}\ln z, \tag{10}$$

where Γ designates the *circulation*. The potential becomes

$$\varphi=-\frac{\Gamma}{2\pi}\tan^{-1}\frac{y}{x}. \tag{11}$$

The function $\tan^{-1}\frac{y}{x}$ is multi-valued; the potential increases by the value Γ each time we travel around the vortex-line. One may free oneself from this multi-valuedness by cutting a branch-cut from the origin to the outside. This branch-cut must not be traversed. (Compare Part I, Sect. E, Art. 8.)

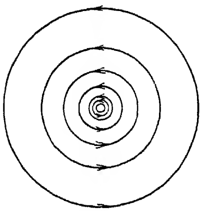


Fig. 49. Flow around a vortex-line.

The stream function is:

$$\psi = -\frac{\Gamma}{4\pi} \ln (x^2+y^2). \tag{12}$$

The equipotential lines in this case are rays through the piercing point of the vortex-line with the xy-plane and the stream lines are concentric circles. The flow, thus, circles about the piercing point in the mathematically positive sense, i.e. counterclockwise (Fig. 49).

In fluid dynamics the function $c \ln z$ is used with real as well as with purely imaginary constant c, as will be shown later.

4. The Source- and Sink-Flow.

Referring to Fig. 50, let a source-line of discharge $+q$ be situated at point $z=-a$, while at the point $z=+a$ a "sink-line" of opposite equal strength $-q$ is located. The total quantity of liquid issuing from the source-line is, therefore, consumed by the sink-line. Let it be required to find the shape of the field of flow. Using equation (7), it follows by superposition that

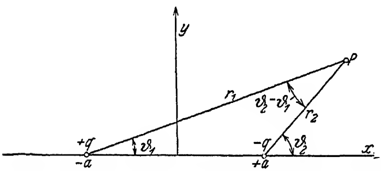


Fig. 50. Referring to the field of flow due to two source-lines.

$$F(z) = \frac{q}{2\pi} \ln (z+a) - \frac{q}{2\pi} \ln (z-a) = \frac{q}{2\pi} \ln \frac{z+a}{z-a} \tag{13}$$

Introducing a pair of polar coördinates with the origin at $z=-a$ and $z=+a$, we have with the notations as in Fig. 50,

$$F(z) = \frac{q}{2\pi} \ln \frac{r_1 e^{i\vartheta_1}}{r_2 e^{i\vartheta_2}} \tag{14}$$

so that the stream function (as imaginary part of equation (14)) becomes:

$$\psi = \frac{q}{2\pi}(\vartheta_1-\vartheta_2) = -\frac{q}{2\pi}(\vartheta_2-\vartheta_1). \tag{15}$$

But the angle $\vartheta_2-\vartheta_1$ is equal to the angle ϑ between r_1 and r_2, and hence:

$$\psi=-\frac{q}{2\pi}\vartheta. \tag{16}$$

Accordingly, the curves $\psi=\text{const}$ are circles through $z=+a$ and $z=-a$, containing ϑ as periphery angle (Fig. 51).

5. Source- and Sink-Flow with Parallel Flow.

Let us superimpose a parallel flow of velocity v_0 (in direction of the positive x-axis) and the above determined field of flow. The complex potential of such a flow is $F_1(z)=v_0z$, of which equation we may convince ourselves by splitting up the potential:

$$\varphi=v_0x,$$

and, hence, by equation (5),

$$v=\frac{\partial\varphi}{\partial x}=v_0.$$

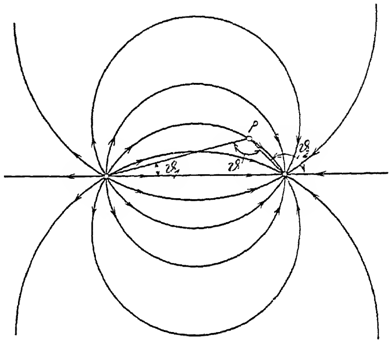


Fig. 51. Source- and sink-flow.

Accordingly, the total field of a source- and sink-line and a parallel field can be described by the following complex potential:

$$F(z)=v_0z+\frac{q}{2\pi}\ln\frac{z+a}{z-a}. \tag{17}$$

The decomposition into real and imaginary part furnishes:

$$F(z)=v_0x+iv_0y+\frac{q}{2\pi}\ln\frac{r_1}{r_2}-\frac{q}{2\pi}i\vartheta. \tag{18}$$

For the stream function we get

$$\psi=v_0y-\frac{q}{2\pi}\vartheta.$$

The stream lines are therefore given by equation

$$v_0y-\frac{q}{2\pi}\vartheta=\text{const}. \tag{19}$$

The one particular stream line, for which the constant is equal to zero, is of special interest. For this stream line we get the x-axis and a closed curve of oval shape enclosing the source- and sink-line (Fig. 52). In a stationary flow the particles of liquid follow the stream lines and no flow exists perpendicular to a stream line; therefore, any stream line may be considered to represent a fixed wall. If we choose the oval as a fixed wall, the field of flow in Fig. 52 represents the plane irrotational flow around a cylinder of oval cross-section. Let us now establish the points of the field at which the velocity has the value zero. Such points are called *stagnation points.* The absolute value of the velocity is given by

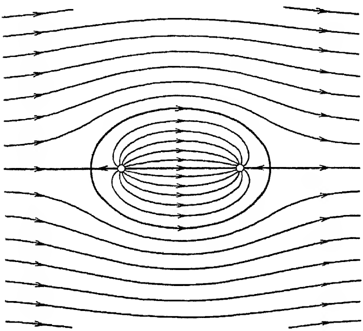


Fig. 52. Flow around an oval cylinder.

$$\frac{dF(z)}{dz}$$

Thus, for stagnation points

$$\frac{dF(z)}{dz} = 0. \tag{20}$$

Differentiating (17), we obtain for the locus vector z_0 of the stagnation point:

$$\frac{dF(z)}{dz}_{=z_0} = v_0 + \frac{q}{2\pi}\left(\frac{1}{z_0+a} - \frac{1}{z_0-a}\right) = 0, \tag{21}$$

or, solving:

$$\frac{qa}{\pi v_0} \tag{22}$$

There thus exist two points on the x-axis, symmetrical with respect to the origin of the system of coördinates, at which points the velocity is zero:

$$z_0 = \pm\sqrt{a^2 + \frac{qa}{\pi v_0}}. \tag{23}$$

Hereby, the half length $A = z_0$ of the oval is given. To determine the half

width B, note that for the equation of the oval the constant in equation (19) is zero, hence

$$r_0 y - \frac{q}{2\pi}\vartheta = 0,$$

from which, for $y = B$, follows:

$$\vartheta = {}^{2\pi r_0}B.$$

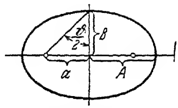


Fig. 53. Determination of the oval.

Referring to Fig. 53, we have

$$\tan\frac{\vartheta}{2} = \frac{a}{B}.$$

For B we, therefore, obtain the transcendental equation:

$$\tan\left(\frac{\pi r_0 B}{q}\right) = \frac{a}{B} = \left(\frac{q}{\pi r_0 B}\right)\cdot\ a\pi r_0 \tag{24}$$

6. Flow Around a Circle.

Let the source- and sink-lines situated at the point $z = \pm a$ move closer and closer to the origin of the system of coördinates, or in other words, let us go over to the limit $a \rightarrow 0$. At the same time let us dispose of the discharge of the source-line such as to keep the "momentum" $M = q \cdot 2a$ constant. Expanding into series,

$$\ln(z+a) = \ln\left\{z\left(1+\frac{a}{z}\right)\right\} = \ln z + \frac{a}{z} - \frac{a^2}{2z^2} + \frac{a^3}{3z^3} - \cdots,$$

$$\ln(z-a) = \ln\left\{z\left(1-\frac{a}{z}\right)\right\} = \ln z - \frac{a}{z} - \frac{a^2}{2z^2} - \frac{a^3}{3z^3} - \cdots$$

and obtaining the difference, we get:

$$\ln\frac{z+a}{z-a} = \frac{2a}{z} + \frac{2a^3}{3z^3} + \cdot \tag{25}$$

Introducing the momentum, we thus have:

$$\lim_{a\rightarrow 0} F(z) = \lim_{a\rightarrow 0}\frac{q}{2\pi}\ln\frac{z+a}{z-a} = \frac{M}{2\pi}\cdot\frac{1}{z}$$

As

$$x+iy \quad \frac{x-iy}{x^2+y^2} \tag{26}$$

the stream lines are given by the following equation:

$$\psi = -\frac{My}{2\pi(x^2+y^2)} = \text{const.} \qquad (27)$$

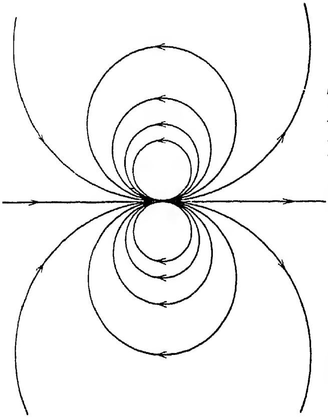


Fig. 54. Double-source.

The stream lines represent circles touching the x-axis at the origin and having their centers located on the y-axis (Fig. 54). Such a flow, produced by two directly adjacent source- and sink-lines, is said tò be a two-dimensional *double-source* or a *dipole* of momentum M.

Let us now superimpose a parallel flow of velocity v_0 (parallel to the negative x-axis) on such a double-source. Thus,

$$F(z) = -v_0\left(z + \frac{M}{2\pi}\cdot\frac{1}{z}\right). \qquad (28)$$

To study the picture of the flow we introduce polar coördinates:

$$F(z) = -v_0\left(re^{i\vartheta} + \frac{M}{2\pi r}e^{-i\vartheta}\right), \qquad (29)$$

or, as

$$e^{i\vartheta} = \cos\vartheta + i\sin\vartheta \text{ and } e^{-i\vartheta} = \cos\vartheta - i\sin\vartheta,$$

$$F(z) = -v_0\left[\left(r + \frac{M}{2\pi r}\right)\cos\vartheta + i\left(r - \frac{M}{2\pi r}\right)\sin\vartheta\right]. \qquad (30)$$

The stream lines are the curves

$$\psi = v_0\left(r - \frac{M}{2\pi r}\right)\sin\vartheta = \text{const.} \qquad (31)$$

Putting the constant equal to zero, we have:

$$v_0\left(r - \frac{M}{2\pi r}\right)\sin\vartheta = 0. \qquad (32)$$

From the above equation it follows that either $\sin\vartheta = 0$, i.e. $\vartheta = 0$ and

$\vartheta=\pi$ (the x-axis of the system of coördinates), or

$$r-\frac{M}{2\pi r}=0,$$

$$r=\sqrt{\frac{M}{2\pi}}=R. \qquad (33)$$

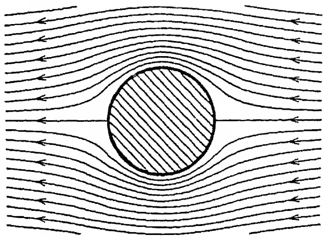


Fig. 55. Eluding flow around a circle.

Thus, the circle about the origin with the radius R is likewise a stream line. If the radius of the cylinder in the path of the flow is prescribed, equation (33) indicates the momentum which has to be inserted at the origin to produce the required picture of the flow. The field of flow determined by equation (31) is indicated in Fig. 55. Such a flow around a circle is said to be an "eluding flow."

For the stagnation points we obtain:

$$\left|\frac{dF(z)}{dz}\right|_{z=z_0}=\left|-v_0\left(1-\frac{R^2}{z_0{}^2}\right)\right|=0, \qquad (34)$$

$$z_0=\pm R.$$

Thus, the points of intersection of the circle with the x-axis are stagnation points. Of further interest is the absolute value of the velocity at the point of intersection of the circle with the y-axis. With $z=\pm iR$, we obtain:

$$|F'(z)|=\left|-v_0\left(1-\frac{R^2}{i^2R^2}\right)\right|=|2v_0|. \qquad (35)$$

The velocity at these points is, therefore, equal to the double velocity of the oncoming flow. It may be shown that this velocity also is the highest velocity appearing in the field of flow outside of the circle.

7. Flow with Circulation Around a Circular Cylinder.

Equation (28) determines the flow around a circular cylinder, the x-axis forming the symmetry line of the flow. Another possible flow around a circle was given by

$$F(z)=-\frac{i\Gamma}{2\pi}\ln z, \qquad (36)$$

this flow representing a rotational symmetric flow about the origin (compare Art. 3). Superimposing equations (36) and (28), we obtain, prescribing the velocity v_0 at infinity, the most general flow of this kind

which is possible around a circular cylinder. We are, thus, looking for a flow with the following boundary conditions:

1. The cylinder wall, $r=R$, should remain a stream line.
2. The flow should go over in a parallel flow in infinity.

The stream lines of equation (36) are concentric circles about the origin. There will exist one stream line coinciding with the circle given by equation (28). Furthermore, the absolute value of the velocity of equation (36)

$$|F'(z)| = \left|\frac{i\Gamma}{2\pi}\,\frac{1}{z}\right| \tag{37}$$

vanishes for $z\longrightarrow\infty$, the prescribed boundary condition thus being satisfied. Investigating the field of flow of

$$F(z) = -v_0\left(z+\frac{R^2}{z}\right)-\frac{i\Gamma}{2\pi}\ln z, \tag{38}$$

we obtain the stream lines

$$\psi=\text{const}=-v_0y\left(1-\frac{R^2}{x^2+y^2}\right)-\frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2} \tag{39}$$

and the absolute value of velocity

$$|F'(z)| = \quad -v_0\left(1-\frac{R^2}{z^2}\right)-\frac{i\Gamma}{2\pi}\cdot\frac{1}{z} \tag{40}$$

The stagnation points z_0 follow from $|F'(z)|_{z=z_0}=0$ after multiplication by z_0^2:

$$v_0(z_0^2-R^2)+\frac{i\Gamma}{2\pi}\cdot z_0=0, \tag{41}$$

and, hence,

$$z_0=-\frac{i\Gamma}{4\pi v_0}\pm\frac{1}{4\pi v_0}\sqrt{16\pi^2v_0^2R^2-\Gamma^2}. \tag{42}$$

Several cases have to be distinguished concerning the location of these stagnation points. Let first $\Gamma<4\pi v_0R$. Then, the absolute value of equation (42) is

$$|z_0|=R, \tag{43}$$

while the argument ϑ has the dual value:

$$\begin{aligned}\vartheta_1&=-\tan^{-1}\frac{\Gamma}{\sqrt{16\pi^2v_0^2R^2-\Gamma^2}}\\ \vartheta_2&=+\tan^{-1}\frac{}{\sqrt{16\pi^2v_0^2R^2-\Gamma^2}}\end{aligned} \tag{44}$$

If Γ changes from zero to $4\pi v_0R$, the stagnation points travel (from their eluding flow position) down on the periphery of the circle till

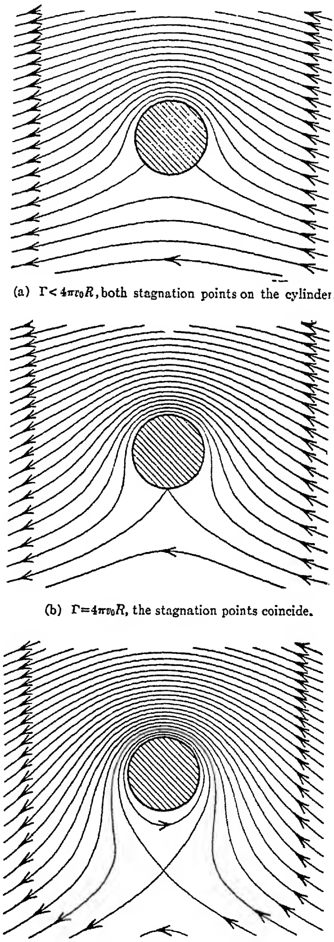
they reunite to a double point at $z_0=-iR$. But if $\Gamma>4\pi v_0 R$, the root expression of equation (42) becomes imaginary, so that one of the stagnation points moves along the imaginary axis into the inside of the field of flow, the other one moving into the inside of the cylinder, thus losing its physical meaning. The flow itself is indicated in Fig. 56 a, b, c, for different values of Γ. We note that the flow no longer proceeds symmetrically to the x-axis, but that the velocity is increased on the upper vertex of the circle and decreased on the lower vertex, because for $z=+iR$ we have

$$|F'(z)|=\left|-2v_0-\frac{\Gamma}{2\pi R}\right| \quad (45)$$

and for $z=-iR$,

$$|F'(z)|=\left|-2v_0+\frac{\Gamma}{2\pi R}\right| \quad (46)$$

We shall demonstrate later that, by virtue of this non-symmetry, a "lift" is created.



(a) $\Gamma<4\pi v_0 R$, both stagnation points on the cylinder.

(b) $\Gamma=4\pi v_0 R$, the stagnation points coincide.

(c) $\Gamma>4\pi v_0 R$, stagnation point in the liquid.

Fig. 56 a, b, c. Flow with circulation around a cylinder (after M. Lagally, "Handbuch der Physik," Vol. 7, Chapter 1).

8. Bernoulli's Equation and Kutta-Joukowsky's Theorem.

For further applications of function theory to two-dimensional fields of flow, let us inspect the following theorems of hydrodynamics:

(a) *Bernoulli's pressure equation.* If v denotes the velocity, p the pressure and ρ the density of the liquid, we have for a stationary flow without the action of external forces

$$p+\frac{\rho v^2}{2}=p_0=\text{const}, \quad (47)$$

where p_0 represents the highest pressure appearing in the total

process of flow. This pressure is obtained by flowing around solid bodies at the stagnation point ($v=0$). Bernoulli's equation (47) further reveals that at points of high velocity there exists low pressure and vice versa. The general flow with circulation around a circular cylinder, as given by equation (38), thus possesses a force acting upon the cylinder in the direction of the positive y-axis as, by equations (45) and (46), the velocity at the upper half of the cylinder is higher than that at the lower one.

In contrast to the above, Bernoulli's equation does not yield a resulting force upon the cylinder (by reasons of symmetry). This applies for the eluding flow (equation 30) as well as for the circulation flow. This result, contradicting the intuitive conception, is called the hydro-dynamic paradoxon. In reality, the always observable resistance is caused by friction.

(b) *The Kutta-Joukowsky Formula.* By means of Bernoulli's equation we have qualitatively realized that a force is acting upon a circular cylinder dipped into a parallel flow with circulatoric motion around the cylinder. This holds in general for any arbitrary contour of the cylinder. By Kutta-Joukowsky's theorem, the magnitude of this force A is determined as follows:

$$A=\rho v_0 \Gamma l. \tag{48}$$

Here, v_0 denotes the velocity of flow in infinity, Γ the circulation, l the length of the cylinder and ρ the density. The force A is called the lift and acts at right angles to the direction of v_0. Kutta-Joukowsky's theorem holds independent of the contour of the cylinder and is basic for the determination of the lift of infinitely long airfoils. We shall take no account of the proof of the theorem, only stating that it follows by application of the impulse theorem.* We shall only refer to an analogy of Kutta-Joukowsky's theorem in electro-dynamics: given the magnetic density lines of a magnetic field of strength B. Let a wire carrying the current I be located perpendicularly to that field. Then, a force P, at right angles to B and I, acts upon the wire of length l, the magnitude of this force being given by the following equation

$$P=BIl. \tag{49}$$

Thus, comparing equations (48) and (49), we see that B corresponds to v_0 and I to the circulation Γ.

* Compare the textbooks referred to at the end of this chapter. Also compare: R. V. Mises, "Zur Theorie des Tragflächenauftriebes," *Zeitschr. Flugtechn*, Heft 21/22 (1917), p. 157.

9. Conformal Mapping of Fields of Flow.

For the circular cylinder we were able to construct the field of flow in a simple manner by superposition of a parallel flow and a double source-line. By adding a rotational symmetric circulatoric flow, we obtained the most general possible two-dimensional flow around a circular cylinder. With the help of the conformal mapping, it is now possible to ascertain the field of flow around other cylindrical bodies as well. Let us assume that the flow around the circle of radius R be prescribed by the function $F(z)$ in the z-plane $(z=x+iy)$. We now map the z-plane upon another plane, the ζ-plane with the coördinates $\zeta=\xi+i\eta$, performing this with the help of a function $\zeta=g(z)$ or, solved for z, $z=h(\zeta)$. In this mapping let the circle in the z-plane go over into the required contour. The required field of flow around the contour is then given by the imaginary part of the function $F\{h(\zeta)\}$.

10. Conformal Mapping $\dfrac{\zeta-2a}{\zeta+2a}=\left(\dfrac{z-a}{z+a}\right)^2$

As an example, consider the conformal mapping of the z-plane upon the ζ-plane, as given by the function

$$\frac{\zeta-2a}{\zeta+2a}=\left(\frac{z-a}{z+a}\right)^2. \tag{50}$$

We investigate: into what curve does a circle lying in the z-plane and going through the points $A(-a,\,0)$ and $B(+a,\,0)$ (located on the real

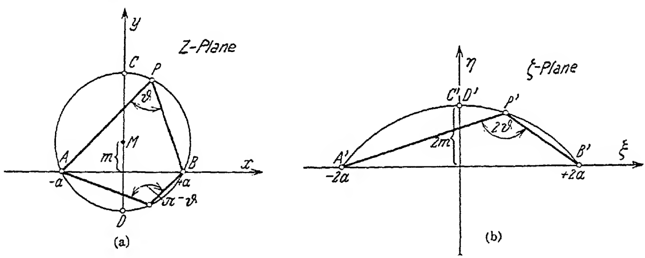


Fig. 57 a and b. Mapping of a circle in the z-plane upon the ζ-plane.

axis) go over? Let the center of the circle M be located on the imaginary axis at a distance m from the origin (Fig. 57a). Solving equation (50) for ζ, we have:

$$\zeta=z+\frac{a^2}{z}. \tag{51}$$

The points A and B are thus mapped into the points A' $(-2a, 0)$ and $B'(+2a, 0)$. The two intersection points of the circle with the imaginary axis, C and D, map, however, into the one Point C' $(0, 2\,mi)$, because for $z=i(m\pm\sqrt{m^2+a^2})$ we have:

$$\zeta=\xi+i\eta=i\left(m\pm\sqrt{m^2+a^2}-\frac{a^2}{m\pm\sqrt{m^2+a^2}}\right)=+2mi.$$

Taking the logarithms on both sides of equation (50), we get

$$\ln\frac{\zeta-2a}{\zeta+2a}=2\ln\frac{z-a}{z+a}.$$

Referring to Art. 4, we see that the imaginary part of $\ln\frac{z-a}{z+a}$ is the angle APB and the imaginary part of $\ln\frac{\zeta-2a}{\zeta+2a}$ is the angle $A'P'B'$. We therefore conclude that

$$\sphericalangle APB=2\sphericalangle A'P'B'.$$

If the point P moves along a circle, the periphery angle remains constant. The image of P in the ζ-plane is P', the latter moving along a curve having a constant periphery angle of double magnitude; hence, P' moves likewise along a circle. The circular arc $A'C'B'$ is traversed twice, as the periphery angle beneath the x-axis is equal to $\pi-\vartheta$, the double angle thus equal to $2\pi-2\vartheta$, which is the angle $A'P'B'$. If M moves to the origin of the z-plane, m vanishes, and the circle is mapped into the doubly traversed section of a straight line from $+2a$ to $-2a$.

11. Flow Around a Plane Plate.

Let us first consider the simplest case, namely that the curve section in the ζ-plane is a straight line $(m=0)$. Then, by equation (51), the circle of radius a is mapped into the straight line of length $4a$ on the ξ-axis. For the field of flow around the circle we found (Art. 7):

$$F(z)=-v_0\left(z+\frac{a^2}{z}\right)-\frac{i\Gamma}{2\pi}\ln z.$$

The velocity v_0 had a direction parallel to the x-axis. If the velocity encloses the angle α with this axis, we have to substitute $ze^{i\alpha}$ for z, hence obtaining:

$$\begin{aligned}F(z)&=-v_0\left(ze^{i\alpha}+\frac{a^2}{ze^{i\alpha}}\right)-\frac{i\Gamma}{2\pi}\ln z\,e^{i\alpha}\\&=-v_0\left(ze^{i\alpha}+\frac{a^2}{ze^{i\alpha}}\right)-\frac{i\Gamma}{2\pi}\ln z-\frac{i\Gamma}{2\pi}\cdot i\alpha.\end{aligned}\tag{52}$$

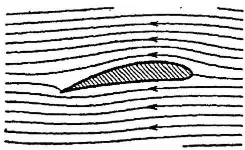
It should be noted, however, that a constant is of no consequence in a flow potential, and that we therefore may omit the last term of equation (52). Hence,

$$F(z)=-v_0\left(ze^{i\alpha}+\frac{a^2}{ze^{i\alpha}}\right)-\frac{i\Gamma}{2\pi}\ln z. \tag{53}$$

We obtain the field of flow in the ζ-plane, by expressing z by equation (51) and then substituting in equation (53). Taking the imaginary part of the thus obtained equation, we find the required field of flow around a plane plate streaming along from infinity under an angle α.

Let us calculate the absolute value of the velocity v at the plate boundaries, that is for $\zeta=\pm 2a$:

$$|v|=\left|\frac{dF}{d\zeta}\right|=\left|\frac{dF}{dz}\cdot\frac{dz}{d\zeta}\right|=\left|\frac{\dfrac{dF(z)}{dz}}{\dfrac{d\zeta}{dz}}\right| \tag{54}$$



(a) Pure irrotational flow.

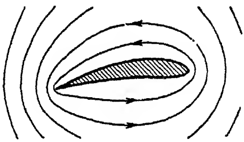
By equation (53), we have:

$$\frac{dF}{dz}=-v_0\left(e^{i\alpha}-\frac{a^2}{z^2e^{i\alpha}}\right)-\frac{i\Gamma}{2\pi z}. \tag{55}$$

With arbitrarily prescribed circulation Γ, this expression is finite everywhere on the boundary of the circle and outside of it.

Furthermore, from equation (51) we get:

$$\frac{d\zeta}{dz}=1-\frac{a^2}{} \tag{56}$$



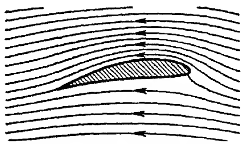
(b) Circulation flow.

thus, for $z=\pm a$, i.e. for those points of the z-plane which go over into the end-points of the straight line section in the ζ-plane,

$$\frac{d\zeta}{dz}=0.$$

Hence

$$\frac{dF}{d\zeta}=\infty$$



(c) Pure irrotational flow with circulation.

Fig. 58 a, b, c. Flow around an airfoil profile.

in equation (54), i.e. the plate edges are flown around with infinite velocity.

12. The Kutta Condition.

Mathematically, infinite velocities always appear if the flow occurs around edges or corners. Physically, such velocities are not possible, as a rupture of continuity of the fluid and the formation of vortices and hollow spaces take place first. We have to distinguish between the leading and trailing edge. With respect to the leading edge, this danger may be eliminated by rounding-off the edge or by increasing its thickness. The experience of aircraft engineering also indicates that airfoils with extra thick front edge show favorable results in flying tests. The rear edge, on the contrary, should remain sharp. The irrotational flow (potential flow), $\Gamma=0$, furnishes at this edge an infinitely high velocity of flow around the edge. To illustrate this, Fig. 58a indicates the irrotational flow around a wing profile; the flowing around the rear edge may be recognized, the rear stagnation point lying on the upper side of the profile.

By a suggestion of Kutta, it is possible to do away with the flow around the rear edge and to assure a smooth flowing-off at this point by putting the rear stagnation point of the flow into the rear edge of the wing profile.

Treating the irrotational flow with circulation around a circle (Art. 7), we have pointed out that the magnitude of circulation Γ may be arbitrarily chosen without infringing the boundary conditions. Based on equation (55), Kutta determines Γ such that $\frac{dF}{dz}$ vanishes at the rear edge:

$$\frac{dF}{dz} = -v_0\left(e^{i\alpha} - \frac{a^2}{z^2 e^{i\alpha}}\right) - \frac{i\Gamma}{2\pi z} = 0 \text{ for } z = -2a. \tag{57}$$

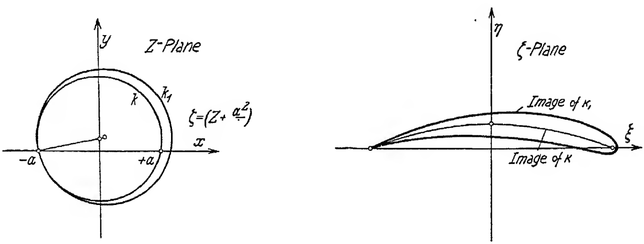


Fig. 59 a and b. Referring to the originating of the Joukowsky profile.

For the velocity, by equation (54), we then obtain the indeterminate expression $\frac{0}{0}$. The limiting case, however, as may be shown by differentiation, remains finite. Equation (57) thus furnishes (as function of

v_0) the value α and an equation for the determination of Γ. Fig. 58b indicates a pure circulation flow around the profile; the strength Γ of the circulation is chosen to assure a smooth flowing-off at the rear edge (Fig. 58c), when superposing Fig. 58a and Fig. 58b. The rear stagnation point coincides with the tip of the profile.

13. Joukowsky Profiles.

The rounding-off of the front edge in order to avoid the infinitely high velocity appearing here may be taken into consideration in a simple mathematical manner by a suggestion of Joukowsky. Referring to the mapping $\zeta=z+\dfrac{a^2}{z}$, the circle k_1 in the z-plane goes over into a circular arc of the ζ-plane (Fig. 59a). If we now try to find the image of another circle k_2 of the z-plane, touching the circle k_1 at the point $-a$ and totally enclosing the circle k_1, we obtain in the ζ-plane a curve, containing the circular arc in the inside and running into a tip at the point $\zeta=-2a$. Fig. 59b indicates that the result is a profile, coming very closely to the ones used in aeronautical engineering.

Only the flow around the circle k_2 is of further interest, this circle going over into the Joukowsky profile. To calculate the flow in a simple manner we refer it (in the z-plane) to a z'-system of coördinates, the origin of which coincides with the center of k_2. By Fig. 60, let M be the center of the circle k_1 in the z-plane, this circle being mapped into the circular arc $A'B'$ of the ζ-plane (by means of the transformation). The center O' of the circle k_2, touching the circle k_1 at point A and the image of which yields the contour of the Joukowsky profile in the ζ-plane, is located on the line AM in a distance d from M. The following geometric relations may be read from the figure:

$R_1=\sqrt{a^2+m^2}$, $R_2=\sqrt{a^2+m^2}+d$

and $m=a\tan\beta$.

The coördinates of O' are:

$$\left.\begin{aligned} x_0&=d\cos\beta \\ y_0&=m+d\sin\beta \\ &=a\tan\beta+d\sin\beta. \end{aligned}\right\} \qquad (58)$$

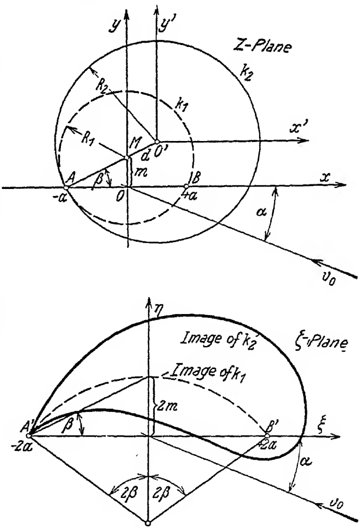


Fig. 60. Referring to the flow around a Joukowsky profile.

We now perform the transformation to a system of coördinates $x'y'$ with the origin O', the x'-axis running parallel to the x-axis and the y'-axis parallel to the y-axis. The complex variable in this system of coördinates is $z'=x'+iy'$. We have:

$$\left.\begin{aligned} x'&=x-d\cos\beta \\ y'&=y-a\tan\beta-d\sin\beta \end{aligned}\right\} \tag{59}$$

The complex variable in the $x'y'$-system becomes

$$\begin{aligned} z'=x'+iy'&=x-d\cos\beta+i\,(y-a\tan\beta-d\sin\beta) \\ &=z-d\cos\beta-i\,(a\tan\beta+d\sin\beta), \end{aligned}$$

or else

$$z'=z-de^{i\beta}-ia\tan\beta.$$

The irrotational flow with circulation around the circle k_2 is given by the imaginary part of

$$F(z')=-v_0\left(z'e^{i\alpha}+\frac{R_2^2}{z'e^{i\alpha}}\right)-\frac{i\Gamma}{2\pi}\ln z', \tag{60}$$

where the velocity v_0 in infinity encloses the angle α with the x-axis. We determine Γ according to Art. 12 such that the velocity v at point A' remains finite. The absolute value of v is

$$|v|=\frac{dF}{d\zeta}=\frac{dF}{dz'}\cdot\frac{dz'}{dz}\cdot\frac{dz}{d\zeta} \tag{61}$$

and should be formed for $\zeta=-2a$, $z=-a$ or $z'=-R_2e^{i\beta}$.

We obtain $\dfrac{dz'}{dz}=1$ and

$$\frac{d\zeta}{dz}=1-\frac{a^2}{z^2}=0 \text{ for } z=-a, \text{ i.e. } \frac{dz}{d\zeta}=\infty.$$

To keep v finite the following relation must exist:

$$\frac{dF}{dz'}=0=-v_0\left\{e^{i\alpha}-\frac{R_2^2}{z'^2e^{i\alpha}}\right\}-\frac{i\Gamma}{2\pi z'}, \text{ for } z'=-R_2e^{i\beta}. \tag{62}$$

This relation is, as previously shown, the determining equation for the magnitude of circulation. Solving for Γ we have:

$$\Gamma=-\frac{2\pi v_0}{i}\left\{z'e^{i\alpha}-\frac{R_2^2}{z'e^{i\alpha}}\right\} \tag{63}$$

or, with $z'=-R_2e^{i\beta}$,

$$\Gamma=\frac{2\pi v_0R_2}{i}\left\{e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}\right\}. \tag{64}$$

As

$$\frac{e^{i(\alpha+\beta)}-e^{-i(\alpha+\beta)}}{2i}=\sin(\alpha+\beta),$$

it follows:

$$\Gamma=4\pi v_0 R_2 \sin(\alpha+\beta).$$

Or, with

$$\left.\begin{aligned} R_2&=\sqrt{a^2+m^2}+d=a\left\{\sqrt{1+\left(\frac{m}{a}\right)^2}+\frac{d}{a}\right\},\\ \Gamma&=4\pi v_0 a\left\{\sqrt{1+\left(\frac{m}{a}\right)^2}+\frac{d}{a}\right\}\sin(\alpha+\beta).\end{aligned}\right\} \tag{65}$$

With the aid of the Kutta-Joukowsky theorem, equation (48), we thus obtain the lift per unit length as follows:

$$A=4\pi\rho\cdot v_0^2 a\left\{\sqrt{1+\left(\frac{m}{a}\right)^2}+\frac{d}{a}\right\}\sin(\alpha+\beta). \tag{66}$$

As may be seen from the calculation, the Joukowsky profiles are marked by two parameters, viz. the ratio $\frac{d}{a}$, which is a criterion of the *thickness*, and the ratio $\frac{m}{a}$, indicating the *convexity* of the profile. Fig. 61 shows a number of such profiles, the contour of which has been obtained by a graphic process suggested by E. Trefftz.*

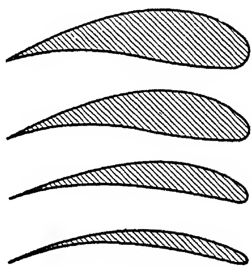


Fig. 61. Joukowsky profiles.

For $d=0$ we get by equation (66) the lift of a circular arc plate:

$$A=4\ \rho v_0^2 a\sqrt{1+\left(\frac{m}{a}\right)^2}\sin(\alpha+\beta), \tag{67}$$

where $4a$ is the length of chord and $2m$ the height of the plate. If also $m=0$, we obtain the lift of a plane plate of width $4a$ as follows:

$$A=4\pi\rho v_0^2 a\sin\alpha. \tag{68}$$

* E. Trefftz, *Zeitschr. Flugtechn*, Vol. 4 (1913), p. 130 and R. V. Mises, *Zeitschr. Flugtechn*, Vol. 11 (1920), pp. 68, 87.

14. Two-Dimensional Flow with Jet Formation.

With the aid of function theory such plane flows in which *free* surfaces appear, so that a formation of jets occurs, can also be treated. As a matter of fact we then have to assume that no external forces are exerted on the liquid, not even the force of gravity. Under this assumption it follows from Bernoulli's pressure equation that the free surface of the liquid is a stream line along which the velocity is constant.

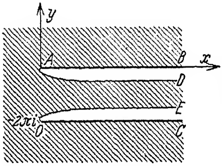


Fig. 62. Flow in a channel.

Let us treat, for example, the flowing-in process of a liquid into a channel $ABOC$ with parallel boundaries from out of an unlimited space. (See Fig. 62.) We have previously interpreted the complex potential χ as function of the complex variable $z=x+iy$, writing

$$\chi(z)=\varphi+i\psi. \tag{69}$$

We may as well consider z as function of χ, thus

$$z=x+iy=z(\chi). \tag{70}$$

The Cauchy-Riemann equations then are:

$$\frac{\partial x}{\partial \varphi}=\frac{\partial y}{\partial \psi}$$

$$\frac{\partial x}{\partial \psi}=-\frac{\partial y}{\partial \varphi}.$$

We had

$$\frac{d\chi}{dz}=\frac{\partial \varphi}{\partial x}+i\frac{\partial \psi}{\partial x}=u-iv;$$

denoting $\frac{dz}{d\chi}$ by ζ, we have:

$$\zeta=\frac{dz}{d\chi}=\frac{1}{u-iv}=\frac{u+iv}{u^2+v^2}=\frac{u+iv}{|\mathbf{v}|^2}, \tag{71}$$

or in polar coördinates

$$\zeta=\frac{dz}{d\chi}=\left|\frac{1}{\mathbf{v}}\right|e^{i\vartheta}. \tag{72}$$

In the above equation, ζ being considered as function of $\mathbf{v}$, ϑ represents the argument of $\mathbf{v}$. The vector ζ thus coincides with the direction of velocity and its magnitude is equal to the reciprocal value of the velocity. We therefore may designate the curve, traversed by the end point of the vector ζ moving along a stream line, as

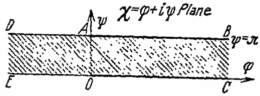


Fig. 63. Region of the complex velocity potential.

a *hodograph*. Let us first consider the complex ζ-plane with the coördinates ξ, η. In this plane, evidently, the free jet boundary (by virtue of the above mentioned physical condition) appears as a *circle* or circular arc, because with $|\mathrm{v}| = \text{const}$ also $\left|\frac{1}{\mathrm{v}}\right|$ must, of course, be constant. Furthermore, we realize from the meaning of ζ that the straight walls in the z-plane are represented by straight lines through the origin of the system of coördinates. In the $\chi = \varphi + i\psi$-plane, the stream lines $\psi = \text{const}$ are straight lines parallel to the φ-axis. One of these straight lines represents the fixed wall as well as the jet boundary. Let this be the line DB in Fig. 63, where the section AB may correspond to the fixed wall, the section AD to the free jet boundary, while EO may represent the other jet boundary and OC the second channel wall. By Fig. 64, the mapping in the ζ-plane is the contour $B'A'D'E'O'C'$, where the channel walls go over into two parallel lines $B'A'$, $O'C'$ of the positive real axis, and the jet boundary goes over into the connecting circle. The required problem is thus solved if the ζ-plane can be mapped upon the χ-plane. Let the result of this mapping be $\zeta = f(\chi)$.

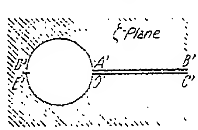


Fig. 64. Hodograph of the channel flow.

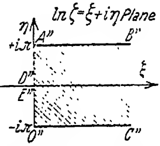


Fig. 65. Mapping in the $w = \ln \zeta$-plane.

Then, from $\zeta = \dfrac{dz}{d\chi}$ it follows by integration:

$$z = \int \zeta \, d\chi = \int f(\chi) d\chi = F(\chi), \tag{73}$$

i.e. we obtain a representation of the required jet boundary as a function of the complex potential.

Starting from here, the carrying out of the mapping is but a problem of function theory. We simplify this problem by mapping the contour in the ζ-plane, by means of the transformation $w = \ln \zeta$, upon the w-plane, at the same time choosing $|\mathrm{v}| = 1$. From equation (72) we follow that

$$w = \ln \zeta = -\ln |\mathrm{v}| + i\vartheta. \tag{74}$$

Accordingly, the circle goes over into the section $A''O''$ (with the length 2π) of the imaginary axis, while the positive real axis of the ζ-plane maps into the two half-rays $A''B''$ and $O''C''$ of the w-plane, as indicated in Fig. 65. But this half-stripe in the w-plane may be mapped upon a full stripe of the χ-plane with the help of Schwarz's procedure. For auxiliary calculations we refer to the next section, specifically treating such map-

ping problems. The mapping function between the χ- and w-plane is found to be

$$w = 2\ln\left(e^{\chi} + \sqrt{e^{2\chi} - 1}\right) - i\pi. \tag{75}$$

This may be verified as follows: To start with, we investigate in what manner the φ-axis maps itself in the w-plane. With $\psi = 0$,

$$w = 2\ln\left(e^{\varphi} + \sqrt{e^{2\varphi} - 1}\right) - i\pi$$

For $0 \leqq \varphi < \infty$, the bracket expression is real and positive. In particular, we obtain for $\varphi = 0$: $w = -i\pi$. Thus, the positive part of the φ-axis is transferred into the straight line section $O''C''$ of the w-plane, lying at a distance $-i\pi$ parallel to the ξ-axis. For $-\infty < \varphi = -|\varphi| \leqq 0$ we have:

$$w = 2\ln\left(e^{-|\varphi|} + \sqrt{e^{-2|\varphi|} - 1}\right) - i\pi = 2\ln\left(e^{-|\varphi|} + i\sqrt{1 - e^{-2|\varphi|}}\right) - i\pi.$$

Splitting up the logarithm into its real and imaginary part, we obtain,

$$\begin{aligned}\ln\left(e^{-|\varphi|} + i\sqrt{1 - e^{-2|\varphi|}}\right) &= \tfrac{1}{2}\ln\left(e^{-2|\varphi|} + 1 - e^{-2|\varphi|}\right) + i\tan^{-1}\frac{1 - e^{-2|\varphi|}}{e^{-|\varphi|}} \\ &= 0 + i\tan^{-1}\left(e^{|\varphi|} - e^{-|\varphi|}\right).\end{aligned} \tag{76}$$

The logarithm thus is purely imaginary, so that the negative φ-axis goes over in the straight line section $O''E''$ of the η-axis of the w-plane.

For the straight line BD of the χ-plane, we have $\chi = \varphi + i\pi$ and thus:

$$w = 2\ln\left(e^{\varphi}\cdot e^{i\pi} + \sqrt{e^{2\varphi}\cdot e^{2\pi i} - 1}\right) - i\pi$$

or, as $e^{i\pi} = -1$ and $e^{2i\pi} = +1$,

$$w = 2\ln\left(-e^{\varphi} + \sqrt{e^{2\varphi} - 1}\right) - i\pi.$$

For $0 \leqq \varphi < \infty$ we obtain:

$$\begin{aligned} w &= 2\ln\left[-1\left(e^{\varphi} - e^{\varphi}\sqrt{1 - \frac{1}{e^{2\varphi}}}\right)\right] - i\pi \\ &= 2\ln(-1) + 2\ln\left[e^{\varphi}\left(1 - \sqrt{1 - \frac{1}{e^{2\varphi}}}\right)\right] - i\pi \\ &= 2\ln\left[e^{\varphi}\left(1 - \sqrt{1 - \frac{1}{e^{2\varphi}}}\right)\right] + i\pi. \end{aligned}$$

The positive part AB of the straight line, hence, goes over in the straight line section $A''B''$ running parallel to the ξ-axis at a distance $+i\pi$, the expression $\ln\left(e^{\varphi}\left(1 - \sqrt{1 - \frac{1}{e^{2\varphi}}}\right)\right)$ being real and positive. For point A, $\varphi = 0$ and thus $w = +i\pi$, i.e. point A'' corresponds to point A.

By virtue of equation (76) the negative part of AD $(-\infty<\varphi\leqq 0)$ maps into the straight line section $A''D''$ of the η-axis.

From equation (75) we derive

$$\zeta=e^{w}=(e^{\chi}+\sqrt{e^{2\chi}-1})^2=2e^{2\chi}+2e^{\chi}\cdot\sqrt{e^{2\chi}-1}-1, \tag{77}$$

and, integrating the above,

$$\begin{aligned} z=x+iy=\int^{\chi}\zeta d\chi &= \int(2e^{2\chi}+2e^{\chi}\sqrt{e^{2\chi}-1}-1)d\chi \\ &= e^{2\chi}-\chi+e^{\chi}\sqrt{e^{2\chi}-1}-\ln(e^{\chi}+\sqrt{e^{2\chi}-1})-1. \end{aligned} \tag{78}$$

The above equation furnishes the jet contour by substitution of the stream line equation $\chi=\varphi+i\pi$, where φ assumes all values from 0 to $-\infty$.

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C. THE FIELD DISTRIBUTION IN THE NEIGHBORHOOD OF EDGES

BY E. WEBER, BROOKLYN

1. The Problem.

In the mathematical part it was shown that any complex analytic function satisfies the potential equation. Accordingly, we may state arbitrarily many solutions of this equation considering any one of such functions, the real or imaginary part of which is, for example, interpreted as the potential of the electrostatic field. In all practical cases, however, the problem is not only to derive *any kind* of a solution of the potential equation, but there always are certain *boundary conditions* to be satisfied. Three different kinds of problems are to be distinguished: if the *potential* at the edge of the region is prescribed we have to deal with a boundary condition problem of first kind; if, on the contrary, if the *derivatives of the potential* at the edge are prescribed — we are facing a boundary condition

problem of second kind. If both conditions alternate in succession along the boundary, the problem is said to be a boundary condition problem of third kind. The theory of functions teaches that all boundary condition problems of first and second kind may be solved by mapping the given region upon a circle. Concerning the application of this "existence theorem," it is of fundamental significance that we can at once obtain the mapping function by means of an integration, provided that the region consists of straight lines forming a polygon. The differential equation of this mapping function is designated as the *Schwarz-Christoffel theorem**(compare Part I, Sect. F, Art. 16).

As far as the problems of this section are concerned, it is sufficient to formulate this theorem for the boundary condition problem of first kind in a *special* manner. Let us consider a polygonally bounded region in the z-plane and let it be required to find the complex potential $\chi=\varphi+i\psi$ as a function of the complex coördinates $z=x+iy$; let furthermore the condition exist that the real and the imaginary parts of the above function assume definite and constant values at the boundaries of the region. The simplest case is the one dealing with two connected boundaries, having different but constant potentials. If we succeed in mapping the given region in the z-plane upon a parallel stripe in the χ-plane, the unknown field of the z-plane is reduced to the known homogeneous field of the χ-plane. Hereby, the function itself satisfies (in the χ-plane) the potential equation and the real or imaginary part of the function possesses a constant boundary-value at the boundaries of the stripe. Now, if $\chi=g(z)$ is the mapping function, the potential equation is satisfied also in the z-plane by this function; moreover, $\chi=g(z)$ satisfies certain boundary conditions on the polygonal contour: those parts of the polygonal contour corresponding to the straight lines $\varphi=\text{const}$ are of constant potential, while the images of the straight lines $\psi=\text{const}$ correspond to the field lines. Thus, the function $\chi=g(z)$ represents the solution of a certain boundary condition problem of simplest kind.

2. Mapping of Polygonal Regions upon the Half-Plane.

The mapping of the χ-plane upon the z-plane often can not be performed in one step. We shall first show that it is sufficient to map the given polygonal region upon the upper half-plane of a complex *auxiliary plane*, the t-plane. We reason as follows: as the rectangular boundary of the homogeneous field in the χ-plane likewise represents a polygonal region, the χ-plane may be mapped upon the t-plane by means of the

* H. A. Schwarz, "Über einige Abbildungsaufgaben," *Crelle's Journ. f. Mathematik*, Vol. 70 (1869), pp. 105–120; Christoffel, "Über die Abbildung einer einfach einblättrigen zusammenhängenden Fläche auf den Kreis," *Göttinger Nachrichten*, (1870).

same process. Eliminating t, the required mapping function $\chi(z)$ is obtained. Schwarz's formula performs the mapping of the polygon upon the upper t-half-plane by means of the following relation:

$$\frac{dz}{dt}=K(t-t_1)^{-\alpha_1}\cdot(t-t_2)^{-\alpha_2}\cdots(t-t_n)^{-\alpha_n}. \tag{1}$$

As this formula has already been derived in the mathematical part of the book (Part I, Sect. F, Art. 16), we content ourselves with the proof of its correctness.

As a matter of clear understanding, let us point out the definitions used in the following discussion: all *angles* shall be counted in the mathematical sense, i.e. positive when counter-clockwise; the *boundary of a region* G shall be understood to represent the limiting line of the region; boundary integrals, i.e. line integrals along the boundary of a region, shall be always taken in such a way as to keep the region G at one's *left side*.

As a preparation, let us treat the case of the prescribed region having but *one corner*. Schwarz's formula then assumes the following simple form:

$$\frac{dz}{dt}=K(t-t_1)^{-\alpha_1}. \tag{2}$$

In the above equation let the variable t move along the real axis of the complex t-plane and let K, as a matter of simplicity, be assumed to be real. Then,

$$(t-t_1)^{-\alpha_1}=|t-t_1|^{-\alpha_1}\cdot e^{-i\alpha_1\arg(t-t_1)}. \tag{3}$$

As long as $t<t_1$ the argument of $(t-t_1)$ has the value π and hence:

$$(t-t_1)^{-\alpha_1}=|t-t_1|^{-\alpha_1}\cdot e^{-i\pi\alpha_1}. \tag{4}$$

As soon as $t>t_1$, however, the argument of $(t-t_1)$ vanishes, and we have

$$(t-t_1)^{-\alpha_1}=|t-t_1|^{-\alpha_1}. \tag{5}$$

From the above we recognize that for $t>t_1$ the element dz falls in the direction of dt, i.e. that the image of that part of the real t-axis lying on the right side of t_1 goes over in a section of the real axis of the z-plane. For $t<t_1$, on the contrary, dz is rotated through the angle $-\gamma_1=-\pi\alpha_1$ against the real axis, so that the image of that section of the real axis of the t-plane on the left side of t_1 becomes a straight line with the slope γ_1 against the real axis of the z-plane. Equation (2) thus performs the mapping of the upper half of the t-plane upon an angular section of the z-plane having an *outside angle* $\gamma_1=\pi\alpha_1$ (Fig. 66).

If the outside angle γ_1 is prescribed, the corresponding exponent α_1 of the differential equation is determined by

$$\alpha_1 = \frac{\gamma_1}{\pi}. \tag{6}$$

Utilizing these results, the general equation (1) may be put in the form

$$\frac{dz}{dt} = K(t-t_1)^{-\frac{\gamma_1}{\pi}} \cdot (t-t_2)^{-\frac{\gamma_2}{\pi}} \cdots (t-t_n)^{-\frac{\gamma_n}{\pi}}. \tag{7}$$

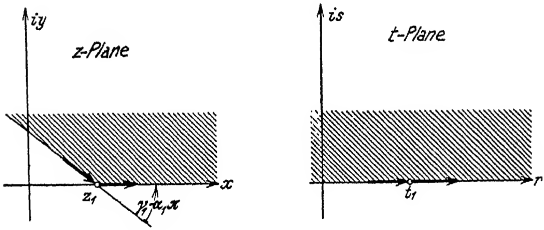

Fig. 66. Mapping of the upper half-plane upon an angular region.

It is sufficient to show that the point $t = t_1$, for example, behaves exactly in the same manner as before. For this purpose let us assume that $t_1 < t_2 < t_3 < \cdots t_n$. Then, for $t < t_1$ all $(t - t_\lambda)$ are negative and, consequently, all $\arg(t - t_\lambda) = \pi$, hence

$$\left.\begin{aligned} &(t-t_1)^{-\frac{\gamma_1}{\pi}} \cdot (t-t_2)^{-\frac{\gamma_2}{\pi}} \cdots (t-t_n)^{-\frac{\gamma_n}{\pi}} \\ &= |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdots |t-t_n|^{-\frac{\gamma_n}{\pi}} \cdot e^{-i\left\{\frac{\gamma_1}{\pi}\arg(t-t_1) + \cdots + \frac{\gamma_n}{\pi}\arg(t-t_n)\right\}} \\ &= |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdots |t-t_n|^{-\frac{\gamma_n}{\pi}} \cdot e^{-i(\gamma_1 + \cdots \gamma_n)} \end{aligned}\right\} \tag{8}$$

If t passes through t_1, only the argument of $(t - t_1)$ changes, and we have:

$$(t-t_1)^{-\frac{\gamma_1}{\pi}} = |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdot e^{-i\frac{\gamma_1}{\pi}\arg(t-t_1)} = |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdot e^{-i\gamma_1} \quad \text{for} \quad t < t_1 \tag{9}$$

$$(t-t_1)^{-\frac{\gamma_1}{\pi}} = |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdot e^{-i\frac{\gamma_1}{\pi}\arg(t-t_1)} = |t-t_1|^{-\frac{\gamma_1}{\pi}} \cdot 1 \quad \text{for} \quad t > t_1 \tag{10}$$

As the element dt is real on both sides of t_1, the corresponding image-elements dz are shifted one against the other by an outside angle γ_1. For all other points t_λ, the same conclusions may be drawn, so that the Schwarz theorem is essentially verified.

Integration of a power generally leads to a power again, the mapping in the neighborhood of a corner thus being furnished in the form of a power function. As in the case of reals, an exception of the above state-

ment is found in the case of a power having the exponent minus one which, when integrated, leads to the logarithm. The exponent $a=+1$ means, from the standpoint of the geometry of mapping, an "angle" $\gamma=+\pi$ (see equation (6)), i.e. a parallel stripe extending into infinity. Choosing the point t_1 in the origin of the system of coördinates, this simple mapping gives (by equation (2))

$$\frac{dz}{dt}=\frac{K}{t} \tag{11}$$

The constant of integration K determines the width b of the parallel stripe. For, carrying out the integration, we must surround the origin by a small circle in the positive real half-plane, this point representing the singular point of the function $\frac{1}{t}$. The integral of the semi-circle (compare Part I, Sect. C, Art. 7) becomes $-\pi i$. This semi-circle integral in the t-plane corresponds to an integration in the regular region of the z-plane extended over the width of the parallel-stripe. Hence,

$$\int dz=b=K\int\frac{dt}{t}=-\pi i K \tag{12}$$

and

$$K=\frac{+b}{-\pi i}=\frac{ib}{\pi} \tag{13}$$

3. Course of Lines of Force at the Edge of a Pole.

Let us treat as a first example of the application of Schwarz's theorem the distribution of lines of force at the edge of a pole of a dynamo (Fig. 67). The field between the surfaces of the pole and the armature is essentially homogeneous, corresponding to the length of the air gap Δ and the difference of magnetic potentials φ_0 of the pole tip surface and the armature surface. At the edges of the poles, however, a distortion of the homogeneous course of lines takes place. Neglecting the curvature of pole tip and armature and considering all radial cross-sections to be equal-valued, we obtain the plane arrangement as indi-

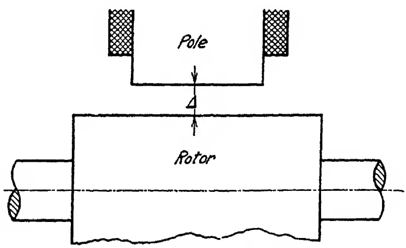

Fig. 67. Schematic cross-section of a dynamo.

cated in Fig. 68. Practically, the pole tip width is always large as compared with the air gap, enabling us to represent the situation at the edge of a pole with sufficient accuracy, as indicated in the simplified Fig. 69. Let us introduce a system of coördinates $z=x+iy$, the origin 0 of which coincides with the corner of the pole tip. The x-axis runs parallel to the boundary of the armature, which latter is given by $y=-\Delta$. Let us, furthermore, designate the point $x=+\infty$ by B and the infinitely remote point of the positive y-axis by A. Then, we should visualize this point as being connected with the infinitely remote point of the negative x-axis by means of a quarter-circle of infinite radius. When performing the mapping upon the t-plane, this quarter-circle is transferred into a single point. At a great distance from the edge the trend of the lines may at once be qualitatively predicted. In the second quadrant they will be quarter-circles, while, between the parallel surfaces, the field is practically homogeneous in a certain distance from the edge. In order to determine the course of lines in the neighborhood of the edge itself let us determine the mapping function of the z-plane upon the t-plane. It appears appropriate to tabulate all the values necessary for the performance of this mapping:

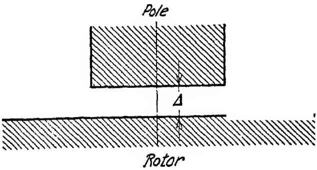

Fig. 68. Simplified cross-section of a dynamo.

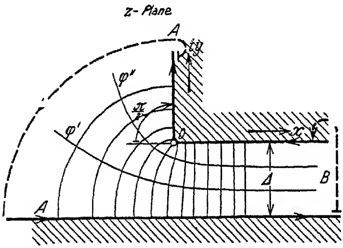

Fig. 69. Schematic representation referring to the mapping problem of the magnetic field of a dynamo.

Point to be mapped z_λ	A	B	O
Image point . . . t_λ	$\pm\infty$	0	$+1$
Outside angle . . $\alpha_\lambda\pi$	$+\frac{3}{2}\pi$	$+\pi$	$-\frac{\pi}{2}$
Exponent . . . . α_λ	$+\frac{3}{2}$	$+1$	1

ılues, we have

$$\frac{dz}{dt}=\frac{C}{(t-0)^{+1}(t-1)^{-\frac{1}{2}}}=C\,\frac{\sqrt{t-1}}{t}. \tag{14}$$

From the two possible values of the root we choose the one going over in $\sqrt{|t|}\cdot e^{\frac{i}{2}\arg t}$ for $t\longrightarrow\infty$.

The point A being the infinitely remote point in the t-plane, does not enter into the mapping function, and this is the reason why the values for A appear in brackets. The shifting of the point B to the origin of the t-plane has the advantage that the two parts of the boundaries of the region, carrying constant potential, go over into the left and right sides of the real axis of the t-plane. The field plot then becomes quite plain. As three points may be chosen arbitrarily, we put O appropriately into such a point as, for example, $t_0=+1$. As two parallel straight lines appear, let us at once evaluate the relation (13). The corner B corresponds to $t=0$, and we, therefore, have by equation (14) and by comparison with equation (11)

$$K=C\sqrt{-1}=iC, \tag{15}$$

and by equation (13), keeping in mind that the line section runs parallel to the positive imaginary axis at the distance Δ, thus $b=i\Delta$, we have:

$$K=iC=\frac{i\Delta}{-i\pi}, \tag{16}$$

$$C=\frac{i\Delta}{\pi}. \tag{17}$$

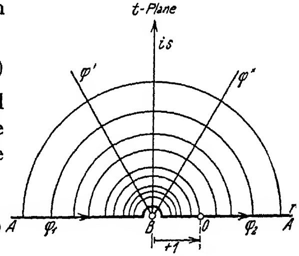

Fig. 70. Field plot in the upper t-plane.

Hereby, the only unknown constant of equation (14) is determined.

The carrying out of the integration does not require any other auxiliary means than those familiar to us from our work with reals. The transformation of the integral

$$\int\frac{\sqrt{t-1}}{t}\,dt\equiv\int\frac{t-1}{t\sqrt{t-1}}\,dt\equiv\int\frac{dt}{\sqrt{t-1}}-\int\frac{dt}{t\sqrt{t-1}} \tag{18}$$

leads to $2\sqrt{t-1}$ for the first term and to $2\tan^{-1}t'$ for the second term if the substitution $(t-1)=t'^2$ is made. Thus we have

$$z=\frac{2\Delta i}{\pi}\left[\sqrt{t-1}-\tan^{-1}\sqrt{t-1}\right]+C_1. \tag{19}$$

C_1 may be determined from the coördination of the point O in Fig. 70. For $t=+1$, z must be equal to $+0$. As the bracket expression becomes zero for $t=1$, C_1 must also be equal to zero and therefore

$$z=\frac{2\Delta i}{\pi}\left[\sqrt{t-1}-\tan^{-1}\sqrt{t-1}\right]. \tag{20}$$

With respect to the multi-valuedness of the appearing functions, it seems recommendable to verify the result. If $t<0$ and real, $(t-1)$ is all the more <0 and it follows:

$$\sqrt{t-1}=i\sqrt{|t|+1}=im; \qquad m>1, \text{ real positive,} \tag{21}$$

so that

$$\tan^{-1}\sqrt{t-1}=\tan^{-1}im=\frac{i}{2}\ln\frac{1+m}{1-m}=+\frac{\pi}{2}-\frac{i}{2}\ln\frac{m-1}{m+1}. \tag{22}$$

Thus,

$$z=\frac{2\Delta i}{\pi}\left[i\left(m-\frac{1}{2}\ln\frac{m+1}{m-1}\right)-\frac{\pi}{2}\right] \tag{23}$$

actually becomes a straight line running parallel to the real axis at the distance $-\Delta$. Moreover,

$$\begin{aligned}&\text{for } |t|=\infty,\ m=\infty \text{ and consequently } z=-\infty-i\Delta,\\ &\text{and for } |t|=0,\quad m=1 \quad \text{and consequently } z=+\infty-i\Delta.\end{aligned}$$

Thus, the image of the armature surface is mapped from the z-plane upon the negative real axis of the t-plane. Furthermore, for values of t, for which $0<t<1$, the line section BO in Fig. 70 is:

$$\sqrt{t-1}=i\sqrt{1-t}=in; \qquad n<1, \text{ real positive,} \tag{24}$$

and hence,

$$\tan^{-1}\sqrt{t-1}=\tan^{-1}in=\frac{i}{2}\ln\frac{1+n}{1-n}. \tag{25}$$

Therefore, by equation (20)

$$z=\frac{2\Delta i}{\pi}\cdot i\cdot\left[n-\frac{1}{2}\ln\frac{1+n}{1-n}\right]=-\frac{2\Delta}{\pi}\left[n-\frac{1}{2}\ln\frac{1+n}{1-n}\right]. \tag{26}$$

But this is the equation of the line section OB of the x-axis.

The point B must be obtainable for $t=0$; for,

$$|t|=0,\ n=1 \text{ and thus } z=+\infty. \tag{27}$$

For the corner O we find:

$$|t|=1,\ n=0 \text{ and thus } z=0. \tag{28}$$

Finally, for real t values ($t>1$) formula (20) holds, the functions having their real meaning. We convince ourselves that z becomes positive and purely imaginary.

The next step is the mapping of the χ-plane of the homogeneous field upon the upper t-plane, the tabulation of the required values being:

Point to be mapped .	χ_ν	A	B	O
Image point	t_ν	$\pm\infty$	0	$+1$
Outside angle	$\alpha_\nu\pi$	$+\pi$	$+\pi$	0
Exponent	α_ν	$+1$	$+1$	0

The mapping function is thus reduced to

$$\frac{d\chi}{dt}=\frac{C'}{(t-0)^1}=\frac{C'}{t}. \tag{29}$$

The distance between the two vertical parallel lines of the χ-plane is $+(\varphi_2-\varphi_1)$ and thus, by equations (13) and (29), we have for $t=0$

$$K=C'=i\frac{\varphi_2-\varphi_1}{\pi}=\frac{i\varphi_0}{\pi}, \tag{30}$$

where $\varphi_0=\varphi_2-\varphi_1$. The integration is carried out without difficulties and, respecting equation (30), yields

$$\chi=i\frac{\varphi_0}{\pi}\ln t+C_1'. \tag{31}$$

The constant C_1' again is obtained by means of an appropriate choice of the point O: for $t=+1$ let $\chi=\varphi_2$. Hence,

$$\chi(1)=C_1'=\varphi_2$$

and thus

$$\chi=i\frac{\varphi_0}{\pi}\ln t+\varphi_2. \tag{32}$$

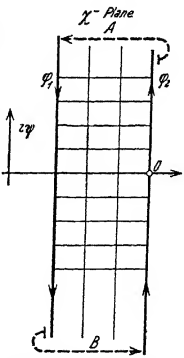

Fig. 71. The homogeneous reference field.

Here too the same validity-test as the one for the mapping of the t-plane may be demonstrated; we shall, however, omit the calculations. Comparing the last equation with equation (11), we find in principle the same structure, equation (32), of course, showing different constants. The lines of force in the t-plane, therefore, are circles about the origin and the lines of potential are radial rays away from the origin. This symmetric and simple picture is the result of the special choice of B in the origin of the t-plane, which choice we already mentioned when mapping the z-plane. The mapping of the χ-plane upon the t-plane appears in all potential problems of similar nature and therefore may always be adopted in the general form of equation (29).

The purpose of the preceding calculation was to obtain the field plot in the originally given plane, the z-plane, by means of mapping of the homogeneous field plot of the χ-plane. We thus have to transfer back the orthogonal net of the χ-plane. From equation (32) we first obtain the auxiliary variable t as a function of the complex potential χ:

$$t=e^{i\pi\frac{\varphi_2-\chi}{\varphi_0}}, \tag{33}$$

or, splitting up into the real and imaginary part,

$$t=r+is=e^{i\pi\frac{\varphi_2-\varphi}{\varphi_0}}\cdot e^{\pi\frac{\psi}{\varphi_0}}=e^{\pi\frac{\psi}{\varphi_0}}\left\{\cos\pi\frac{\varphi_2-\varphi}{\varphi_0}+i\sin\pi\frac{\varphi_2-\varphi}{\varphi_0}\right\} \tag{34}$$

Herewith, any point of the χ-plane or any series of points may be transferred upon the t-plane. The further coördination of the z-points follows from equation (20), cumbersome numeric calculations only appearing for complex values of z.

With the help of equation (20), the second part of the problem, viz. to state the field strength at any point of the z-plane, is now also made possible without any difficulty. The field vector $\mathbf{v}$ is the negative conjugate complex quantity to the differential quotient of the complex potential

$$\mathbf{v}=-\left(\frac{d\chi}{dz}\right)^{\dagger}=-\left[\left(\frac{d\chi}{dt}\right)\cdot\left(\frac{dt}{dz}\right)\right]^{\dagger}$$

the cross indicating the conjugate complex quantity to $\left(\frac{d\chi}{dt}\right)$. With

$$\frac{d\chi}{dt}=\frac{i\varphi_0}{\pi t}; \qquad \frac{dt}{dz}=\frac{\pi t}{\sqrt{t-1}\cdot i\Delta} \tag{35}$$

we have:

$$\mathbf{v}=-\left(\frac{\varphi_0}{\Delta}\frac{1}{\sqrt{t-1}}\right)^{\dagger}. \tag{36}$$

Denoting by

$$\mathbf{v}_\infty=\frac{\varphi_0}{i\Delta}=\left(\frac{\varphi_0}{-i\Delta}\right)^{\dagger} \tag{37}$$

the homogeneous field vector between two parallel horizontal planes having the distance Δ at a potential difference φ_0, we get

$$\frac{\mathbf{v}}{\mathbf{v}_\infty}=\left(\frac{i}{\sqrt{t-1}}\right)^{\dagger}. \tag{38}$$

This form may also be easily checked. Assuming $\varphi_2>\varphi_1$, $\mathbf{v}_\infty$ becomes negative imaginary. If z moves along the boundary of the region, t

follows the r-axis, thus remaining real. Using the simplifications mentioned in connection with the checking of the mapping of the z-plane, we obtain for $t<0$:

$$\frac{\mathbf{v}}{\mathbf{v}_\infty}=\left[\frac{i}{im}\right]^\dagger=\frac{1}{m}. \tag{39}$$

Thus, $\mathbf{v}$ is parallel to the y-axis and is directed downwards, consequently striking upon the potential surface φ_1 at right angles. In a similar manner it follows for $0<t<1$:

$$\frac{\mathbf{v}}{\mathbf{v}_\infty}=\left[\frac{i}{in}\right]^\dagger=\frac{1}{n}. \tag{40}$$

The direction of the field vector is the same, but the magnitude on BO is always larger than $\mathbf{v}_\infty$ (because of $n<1$), while the magnitude on AB was always smaller than $\mathbf{v}_\infty$ (because of $m>1$). The closer t approaches O, the more n moves towards zero, the magnitude of the field vector growing larger and larger, until it grows beyond all limits at the edge itself. We thus realize that a concentration of the field takes place at the edge. If finally $t>1$,

$$\frac{\mathbf{v}}{\mathbf{v}_\infty}=\left[\frac{i}{\sqrt{t-1}}\right]^\dagger=-\frac{i}{\sqrt{t-1}}. \tag{41}$$

The magnitude of the field vector decreases again and, eventually, for $t=\infty$, attains the value zero. Fig. 71 indicates the course of the field vector along the boundary surface with the potential $\varphi=\varphi_2$, whereby the boundary BOA has been developed into a straight line without distortion of scale (for the purpose of simplest representation).

At the edge itself, $t=1+i\epsilon$, where ϵ moves towards zero. From equation (38) we follow:

$$\frac{\mathbf{v}}{\mathbf{v}_\infty}=\left[\frac{i}{\sqrt{i\epsilon}}\right]^\dagger=\left[\frac{1+i}{\sqrt{2\epsilon}}\right]^\dagger=\frac{1-i}{\sqrt{2\epsilon}}. \tag{42}$$

Thus $\mathbf{v}$ becomes infinitely large at the limit $\epsilon=0$ and in direction of the bisect between both surfaces of the corner. This statement holds for all edges showing flat inside angles. As may be easily realized, the magnitude of the field vector is always zero at edges with pointed inside angles. Practically, such an infinitely sharp edge never exists, there always existing a certain rounding-off (if even with but a very small radius). It therefore is of interest to study the influence of a very small rounding-off. The exact mapping of a region with circularly rounded-off corner is possible, but the problem leads to a non-linear differential equation of second degree,* the solution of which is

* The fundamentals may be found in the work of H. A. Schwarz: See footnote in Part II, Sect. C, Art. 1.

impossible, the development into series being also very cumbersome. A simple approximation method is to draw the equipotential lines $\varphi = \text{const}$ in the closest neighborhood of the edge and to consider them as new boundaries. The fact that we obtain hereby quite useful results is emphasized by Fig. 72, in which two closely adjacent potential lines, computed according to the transfer formulas, are drawn together with the orthogonal field lines. In the neighborhood of the edge the smallest radius of curvature may be considered as a degree of the rounding-off. Calculating the field vector along these potential lines, we can (approximately) plot it in Fig. 73 and realize the strong influence of an ever so small rounding-off.

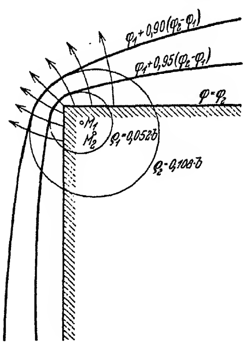

Fig. 72. Potential lines and lines of force in the neighborhood of the edge.

If we calculate the field vector along the above mentioned potential line φ_2' in the neighborhood of the edge, we obtain too small values as compared with the true field vector along $\varphi = \varphi_2$, the potential φ_2' being smaller than φ_2. The magnitude of the field vector should, therefore, be increased by

$$\frac{\varphi_2 - \varphi_1}{\varphi_2 - \varphi_1}$$

4. Transfer of the Field between Edge and Plane to Other Problems.

The preceding problem referred to the field distribution in the neighborhood of a pole edge of a dynamo. Quite similar field forms are met in numerous other problems of electrical engineering. Let us discuss in the following a few *electrostatic fields* of this nature. Thus, for example, the field between the high-voltage winding of a transformer and its grounded core is, in accordance with Fig.

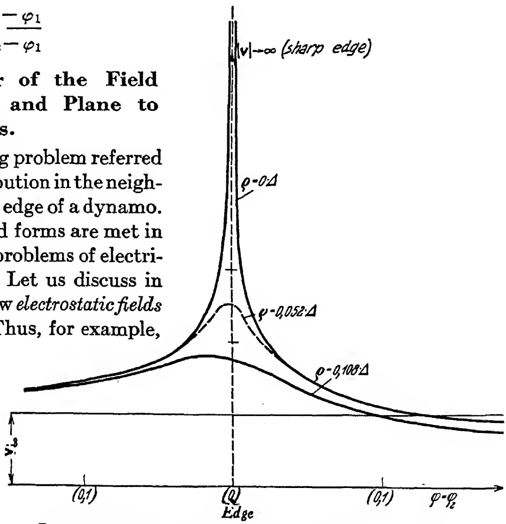

Fig. 73. Distribution of field strength in the neighborhood of an edge for different potential surfaces (see Fig. 72).

74, governed by the same relations, the potential difference between the two contours now being equal to the voltage of the winding against ground. The magnetic field strength is here replaced by the electric field strength $\mathbf{E} = -\text{grad}\,\varphi$ which is responsible for the stress of the insulation material. We also may consider the bodies limited by the surfaces φ_1 and φ_2 to represent electrodes in an electrolyte; we then obtain the *plane parallel electric flow in the electrolyte.* Here, φ_1 and φ_2 are again the potentials, $\varphi_2 - \varphi_1 = \varphi_0$ is the voltage between both electrodes, and the field vector becomes identical with the electric field strength $\mathbf{E}$ or the current density

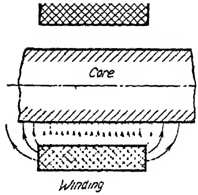

Fig. 74. Electric field of the high-voltage winding of a transformer.

$$\mathbf{j} = \frac{\mathbf{E}}{\rho}. \tag{43}$$

In this Ohm's law ρ is the specific resistance of the electrolyte. As we easily can construct such a plane flow in an electrolyte for any electrode shape, we arrive at a method, which, by reason of the intrinsical relationship between the three physical fields mentioned above, enables us to measure electro-magnetic fields experimentally.*

Another application is the theory of *heat conduction.* Here, the potential is replaced by the temperature T and the field vector by the temperature gradient ($-\text{grad}\,T$), or the temperature drop. We assume the bodies bounded by the surfaces φ_1 and φ_2 to be infinitely good heat conductors. Accordingly, these surfaces carry constant temperatures T_1 and T_2, so that the potential (the temperature) again satisfies the above mentioned boundary condition. Within the separating medium, assumed to be a poor heat conductor, a heat flow free of sources develops, the temperature in steady state satisfying Laplace's equation.

The potential equation also appears in hydro-dynamics in the form of a conditional equation for irrotational motion of an incompressible liquid.† The velocity potential φ satisfies the potential equation and furnishes in well-known manner the velocity. To remain at our example, Fig. 69, the liquid would flow from the surface φ_1 to the surface φ_2; but we also may interchange stream

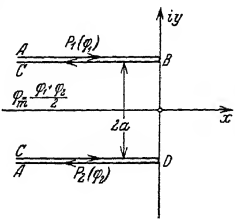

Fig. 75. Schematic representation of a plate condenser.

* J. Labus, "Der Potential-und Feldverlauf längs einer Transformatorenwicklung," *Arch. Elektrot.*, Vol. 21 (1928), p. 250.

† See the contribution by K. Pohlhausen.

lines and potential lines and then obtain essentially the same mapping function of the irrotational flow around an edge.

5. The Plate Condenser.

The first physical application of Schwarz-Christoffel's theorem was due to G. Kirchoff, who investigated the field at the edge of a plate condenser by means of this method. Let each of the two infinitely extended plates P_1 and P_2 be given in the form of a mathematical double-surface (Fig. 75) and let the distance between the plates be $2a$. If the potentials of the plates are φ_1 and φ_2, the plane of symmetry is likewise a potential plane with the potential $\varphi_m=\frac{\varphi_1+\varphi_2}{2}$. It is sufficient to consider the upper half alone. The mapping is carried out according to Figs. 76, 77, 78. The coördination of the points (we may arbitrarily choose all three points) furnishes the differential equation of the required mapping function of the z-plane upon the t-plane:

$$\frac{dz}{dt}=\frac{K}{(t+1)^{-1}\cdot(t+0)^{+1}}=K\frac{t+1}{t} \tag{44}$$

The constant K is determined by equation (13) if we put $t=0$ and note that the distance appears as a quantity pointing vertically down, $b\equiv -ia$. Hence,

$$K=\frac{-ia}{-i\pi}=+\frac{a}{\pi}. \tag{45}$$

Taking this value into consideration, the integration furnishes:

$$z=\frac{a}{\pi}[t+\ln t]+K_1 \tag{46}$$

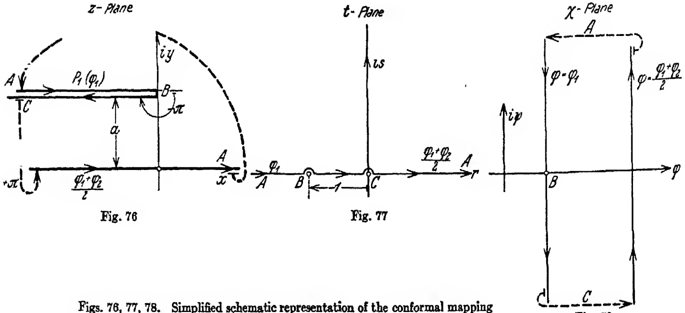

Fig. 76

Fig. 77

Fig. 78

Figs. 76, 77, 78. Simplified schematic representation of the conformal mapping of a plate condenser.

From the position of B we can determine K_1, as for $t=-1$, z should be equal to $+ia$. Thus:

$$ia=\frac{a}{\pi}[-1+i\pi]+K_1; \qquad K_1=\frac{a}{\pi}. \tag{47}$$

The mapping of the t-plane upon the χ-plane is the same as under Art. 3 and consequently equation (31) is preserved, φ_0 standing for $\varphi_m-\varphi_1=\dfrac{\varphi_2-\varphi_1}{2}$.

Therefore,

$$\chi=i\,\frac{\varphi_0}{\pi}\ln t+K_1'. \tag{48}$$

The point $t=-1$ now corresponds to the point $\chi=\varphi_1$ and thus:

$$\chi(-1)=\varphi_1=\frac{i\varphi_0}{\pi}\cdot i\cdot \pi+K_1'; \tag{49}$$

$$K_1'=\frac{\varphi_1+\varphi_2}{2}=\varphi_m. \tag{50}$$

As before, and taking the above values into account, the field strength becomes:

$$\mathbf{E}_z=\left(\frac{d\chi}{dz}\right)^{\dagger}=-\left(\frac{d\chi}{dt}\cdot\frac{dt}{dz}\right)^{\dagger}$$

$$=-\left[\frac{i(\varphi_2-\varphi_1)}{2\pi}\cdot\frac{1}{t}\cdot\frac{t}{t+1}\cdot\frac{\pi}{a}\right]^{\dagger}=\left[\frac{-i}{2a}\,\frac{\varphi_2-\varphi_1}{t+1}\right]^{\dagger}. \tag{51}$$

The checking of all results at the boundaries of the mapping region may be carried out analogously to that in Art. 3. The edge of the upper plane, in particular, corresponds to point $t=-1$, so that the calculation again shows infinitely high field strength at the edge (edge-effect).

The field plot of a plate condenser was first utilized by Rogowski, the purpose having been to develop a measuring spark gap which was supposed not to show any field increase due to edge-effect.

In order to obtain a general picture of the course of the field strength, let us express the auxiliary variable t (by equation (48)) by means of the complex potential χ and, for the sake of brevity, introduce the

complex potential difference against the center line and put $\bar{\chi}=\bar{\varphi}+i\bar{\psi}=\varphi_m-\chi$. Then:

$$|E|=\left|\frac{d\chi}{dz}\right|=\frac{1}{\left|\frac{dz}{d\chi}\right|}=\frac{\pi}{a}\cdot\left|\frac{1}{e^{\pi i\frac{\chi}{\varphi_0}}+1}\right| \tag{52}$$

Introducing $|E_\infty|=\frac{\varphi_0}{a}$ as the homogeneous field strength in infinity and splitting up the exponential function into the real and imaginary part, we have:

$$\left|\frac{E}{E_\infty}\right|=\frac{1}{\sqrt{1+e^{-2\pi\frac{\bar{\psi}}{\varphi_0}}+2e^{-\pi\frac{\bar{\psi}}{\varphi_0}}\cdot\cos\pi\frac{\bar{\varphi}}{\varphi_0}}}. \tag{53}$$

The local dependency of the field strength along an equipotential line is gained by putting $\bar{\varphi}=\text{const}$ for subsequent values of $\bar{\psi}$. Of particular interest is the maximum of field strength. For this purpose we form

$$\frac{\partial\left|\frac{E}{E_\infty}\right|}{\partial\left(\frac{\bar{\psi}}{\varphi_0}\right)}=\frac{-2\pi e^{-2\pi\frac{\bar{\psi}}{\varphi_0}}-2\pi e^{-\pi\frac{\bar{\psi}}{\varphi_0}}\cdot\cos\pi\frac{\bar{\varphi}}{\varphi_0}}{-2\left[1+e^{-2\pi\frac{\bar{\psi}}{\varphi_0}}+2e^{-\pi\frac{\bar{\psi}}{\varphi_0}}\cdot\cos\pi\frac{\bar{\varphi}}{\varphi_0}\right]^{3/2}}=0. \tag{54}$$

From the above, the locus of the maximum field strength follows in the form of the conditional equation

$$e^{-\pi\frac{\bar{\psi}}{\varphi_0}}=-\cos\pi\frac{\bar{\varphi}}{\varphi_0}, \tag{55}$$

or

$$\frac{\pi\bar{\psi}}{\varphi_0}=\ln\frac{1}{-\cos\pi\frac{\bar{\varphi}}{\varphi_0}} \tag{56}$$

From this, real values of $\bar{\psi}$ are obtained only for $\frac{\varphi}{\varphi_0}>\frac{1}{2}$. At the equipotential line,

$$\bar{\varphi}=\frac{\varphi_0}{2}, \tag{57}$$

no field strength maximum appears, the field strength decreasing from the value E_∞ to zero. The contour represented by equation (57) is indicated in Fig. 79 and was chosen by Rogowski as the boundary of his spark gap for break-down studies.

6. The Magnetic Field of a Slot.

For the computation of the field of an electric machine, the knowledge of the air-gap field is of utmost significance. It should be noted that the stator and rotor surfaces in general possess numerous slot openings, so that the magnetic resistance between stator and rotor is essentially higher than the resistance corresponding to the distance between the surfaces. It thus appears necessary to take this increased magnetic resistance into account; this is done by introducing an effective air gap $\Delta_{\text{eff.}} > \Delta$ in place of the actual air gap Δ. The calculation of this effective air gap has, under certain simplifying assumptions, been reduced by Carter to a problem on conformal mapping. To free ourselves from the difficulty of slots both in stator and rotor, we refer our calculations to an appropriately laid mean potential surface, which, in first approximation, may be regarded as a plane (Fig. 80). Then, the slot should be considered as the *one* potential surface φ_0 opposite a smooth iron surface — the above mentioned mean potential surface φ_m. Furthermore, we assume the slot pitch and the slot depth to be infinitely large and thus use the simplified Fig. 81. We realize that it is now possible to represent correctly the field between the edge of the slot and the air gap. Because of the symmetry about the slot centre it is sufficient to consider the right side only. As the polygon side AB itself is a line of force, it is not possible to obtain the simple polar picture of lines of force in the t-plane. Accordingly, the then following mapping upon the χ-plane does not reproduce the whole infinitely extended plate condenser, but only that half of the condenser which lies in the upper half of the χ-plane, and this because the mapping of the section AB of lines of force must again be a line of force. Otherwise the mapping should be treated quite analogously to the previously discussed problem; for, from the mathematical point of view, it does not

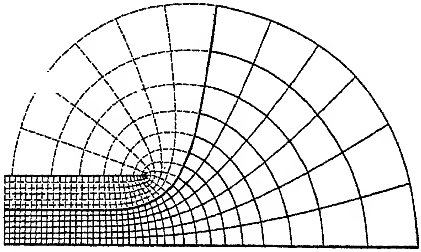

Fig. 79. Development of the measuring spark gap after Rogowski.

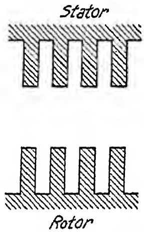

Fig. 80. Schematic cross-section of a dynamo having slots both in stator and rotor.

matter what physical meaning is attached to the boundary line, as long as the mapping itself is chosen correctly. With the help of the drawn-in arrows and angles the derivative of the mapping function for the z-plane results from Schwarz's relation:

$$\frac{dz}{dt}=\frac{K}{(t+a)^{1/2}\cdot t^{1}\cdot (t-1)^{-1/2}}=\frac{K}{t}\sqrt{\frac{t-1}{t+a}} \qquad (58)$$

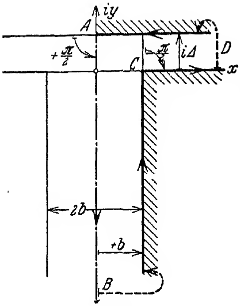

Fig. 81

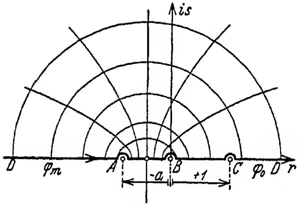

Fig. 82

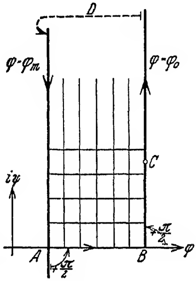

Fig. 83

Figs. 81, 82, 83. Simplified schematic representation for the conformal mapping of the magnetic field of a slot.

As four corner points exist here from which, however, only three may be chosen, a and K remain over as unknown constants. By equation (13) we can determine K, the corner point D lying in infinity in the z-plane. Another relation may be derived from equation (13), pertaining to the corner point B, which relation furnishes the second constant a.

For corner D:*

$$K=-i\frac{i\Delta}{\ }. \qquad (59)$$

For corner B:

$$\frac{iK}{\sqrt{a}}=ib \qquad (60)$$

From both equations, therefore, we get:

$$K=\frac{\Delta}{\pi}; \qquad a=\left(\frac{\Delta}{b}\right)^{2}. \qquad (61)$$

The derivative of the mapping function for the χ-plane is

$$\frac{d\chi}{dt}=\frac{K'}{(t+a)^{1/2}\cdot t^{1/2}}=\frac{K'}{\sqrt{t(t+a)}}, \qquad (62)$$

* As D is the infinitely remote point in the t-plane, equation (13) has to be used with opposite signs, the closing arc, when integrating, being traversed in the reverse sense.

where K' follows by equation (13):

$$K' \quad -i\frac{-(\varphi_0-\varphi_m)}{} \quad ; \varphi_0-\varphi_m \tag{63}$$

The theoretic potential problem is thus solved. As for the applications, we are above all interested in the course of the magnetic densities along the slot walls. The magnetic density being proportional to the field gradient and omitting non-essential constants, we obtain:

$$\mathbf{B}_t = - \frac{\cdot\,\varphi_0-\varphi_m \qquad \dagger}{\dfrac{\pi \qquad \sqrt{(t+a)t}}{\dfrac{\Delta}{\pi t}\sqrt{\dfrac{t-a}{t+a}}}} =_{\imath} \varphi_0-\varphi_m \left[\sqrt{\frac{t}{t-1}}\right]^{\dagger} \tag{64}$$

It is less cumbersome to derive the total flux from the current function, thus avoiding integration. The demonstration of this calculation is of no interest from the standpoint of the application of Schwarz's theorem. We therefore omit this calculation and just refer to the bibliography.

The mapping of more general slot shapes has been later treated by several other authors. In this connection let us indicate briefly the taking into consideration of finite slot depth and slot pitch. Fig. 84 shows a slot of finite depth and pitch. According to the designations of this figure, the derivative of the mapping function for the z-plane becomes

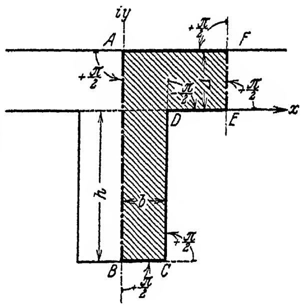

Fig. 84. Schematic representation referring to the conformal mapping of the magnetic field of a slot with finite depth and pitch.

$$\frac{dz}{dt} = \frac{K}{(t+m)^{1/2}(t+n)^{1/2}t^{-1/2}(t-1)^{1/2}(t-p)^{1/2}} = K\sqrt{\frac{t}{(t+m)(t+n)(t-1)(t-p)}}; \tag{65}$$

the mapping function itself is represented by a hyperelliptic integral. In the above formula m, n and p denote constants to be determined after the integration. Therefore, we cannot compute numerically the problem supplemented in the above manner which brings us back to simplifications, for example to the Carter method. The mapping upon the χ-plane also becomes more difficult. As two lines of force had to be considered as parts of the boundary of the region (by reason of sym-

metry), the χ-plane represents a finite section from an infinitely extended plate condenser. The ratio of the sides of the rectangle is already determined by the constants of the z-plane. The mapping function is an elliptic integral, its derivative is:

$$\frac{d\chi}{dt} = \frac{K'}{(t+m)^{\frac{1}{2}}(t-1)^{\frac{1}{2}}(t-p)^{\frac{1}{2}}} = \frac{K'}{\sqrt{(t+m)(t-1)(t+p)}} \qquad (66)$$

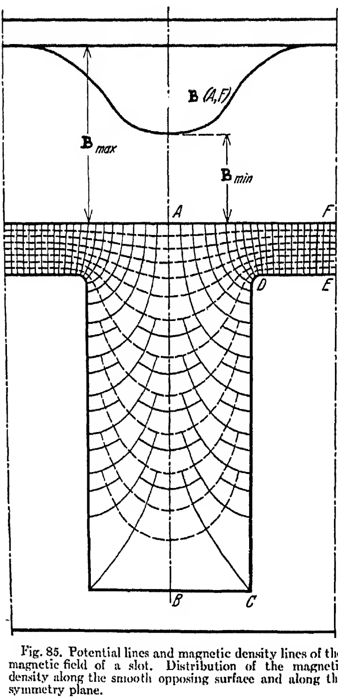

Fig. 85. Potential lines and magnetic density lines of the magnetic field of a slot. Distribution of the magnetic density along the smooth opposing surface and along the symmetry plane.

Fig. 85 indicates the distribution of the magnetic density along the smooth opposing surface and along the symmetry axis of the slot. The potential and field lines are also drawn in, the net being formed in such a way as to assure the traversing of the same magnetic flux between two field lines.

7. Bibliography

The treated examples should be regarded as first references concerning the usefulness of Schwarz's theorem. From the numerous papers written on that subject let us, without claiming the completeness, refer to a few of them.

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D. THE COMPLEX TREATMENT OF ELECTRIC AND THERMAL TRANSIENT PHENOMENA

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1. The Problem.

The treatment of *transient phenomena* by means of the methods of function theory finds its origin in the representation of impulses by complex integrals. An impulse in this connection is understood to be a force suddenly applied to a system in equilibrium. The simplest representation of such a force is an impulse $f(t)$, suddenly rising at the impulse time $t=0$ to its final value 1 and remaining at unity from then on (Fig. 86). Thus:

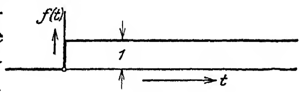

Fig. 86. Constant impulse.

$$\left.\begin{aligned} f(t) &= 0 \text{ for } t<0 \\ f(t) &= 1 \text{ for } t>0 \end{aligned}\right\} \tag{1}$$

the impulse is represented by the following hook-integral:

$$f(t) = \frac{1}{2\pi i} \cdot \int_{-i\infty}^{+i\infty} \frac{e^{pt} dp}{p}. \tag{2}$$

We convince ourselves easily of the correctness of the above expression by using the residue theorem, as was shown in the mathematical part of this book (Part I, Sect. F, Art. 2). This representation is well adapted to be generalized upon other important impulse forms. Let us first consider an impulse of unit magnitude, suddenly starting at the time $t=0$ and suddenly vanishing after the period of action T_w* (Fig. 87). For this impulse we have:

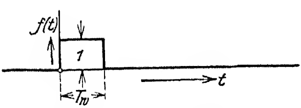

Fig. 87. Temporarily constant impulse.

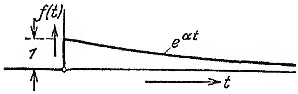

Fig. 88. Exponential impulse ($\alpha < 0$).

$$\left.\begin{aligned} f(t) &= 0 \text{ for } t<0 \\ f(t) &= 1 \text{ for } 0<t<T_w \\ f(t) &= 0 \text{ for } t>T_w \end{aligned}\right\} \tag{3}$$

We may interpret this impulse as a superposition of two impulses of the nature of equation (1), thus writing:

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{e^{pt}dp}{p} - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{e^{p(t-T_w)}dp}{p}. \tag{4}$$

Of further interest is an impulse, suddenly beginning at the time $t=0$ and from here on following an exponential law $e^{\alpha t}$ (Fig. 88). This impulse is given by the hook-integral:

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{e^{pt}dp}{p-\alpha}. \tag{5}$$

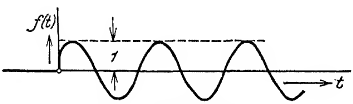

Fig. 89. Oscillating impulse.

It should be noted that in the above equation the real part of α must be assumed to be smaller or equal to zero, an assumption which may be easily verified by means of the residue theorem. Finally, the oscillating impulse of angular velocity ω should be noted; this impulse (Fig. 89) starts discontinuously and is reproduced by the following expression:

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{e^{pt}dp}{p-i\omega}. \tag{6}$$

* Translators remark: The index "w" refers to the German word "Wirkungsdauer" meaning "period of action."

The pole $p=i\omega$ has to be evaded by means of a small semi-circle in the positive real p-plane.

2. Physical Conception of Transient Phenomena in Temperature Fields and in Eddy-Current Fields.

Let us discuss in the following articles a few phenomena of heating processes and eddy-current fields in electric machinery, these phenomena being picked from the vast realm of transient phenomena. We shall obtain the space-time relational equation of such fields by physical reasoning.

The *theory of heating* is based on the conception of a heat current $\mathbf{w}$ flowing in the heated material, the density of the current at any point being proportional to the thermal conductivity λ of the material and the drop of the temperature ϑ, thus:

$$\mathbf{w}=-\lambda\cdot\operatorname{grad}\vartheta. \tag{7}$$

Any stagnation of this flow of heat gives cause to a temperature rise with respect to time according to the specific heat c of the material, thus:

$$\operatorname{div}\mathbf{w}=-c\cdot\frac{\partial\vartheta}{\partial t}, \tag{8}$$

so that if λ is constant in space we have:

$$\operatorname{div}\operatorname{grad}\vartheta=\frac{\partial^2\vartheta}{\partial x^2}+\frac{\partial^2\vartheta}{\partial y^2}+\frac{\partial^2\vartheta}{\partial z^2}=\frac{1}{a^2}\cdot\frac{\partial\vartheta}{\partial t};\quad a^2=\frac{\lambda}{c}. \tag{9}$$

This is the *equation of heat conduction;* the equations of the axial electric eddy field $\mathbf{E}$ in electric machines form an analogy to the above equation. For, from the Faraday Induction Law we can determine the components $\mathbf{H}_x$ and $\mathbf{H}_y$ of the magnet field $\mathbf{H}$ in a material of permeability μ, the relations being:

$$\frac{\partial\mathbf{E}}{\partial x}=-\mu\Pi\frac{\partial\mathbf{H}_y}{\partial t},\qquad\frac{\partial\mathbf{E}}{\partial y}=+\mu\Pi\frac{\partial\mathbf{H}_x}{\partial t},\qquad\Pi=4\pi\cdot10^{-9}. \tag{10}$$

The Ampère Law furnishes the current density $\mathfrak{j}$ for the conductivity κ:

$$\mathfrak{j}=\kappa\cdot\mathbf{E}=\frac{\partial\mathbf{H}_x}{\partial y}-\frac{\partial\mathbf{H}_y}{\partial x}. \tag{11}$$

Consequently, the equation of the eddy field is:

$$\frac{\partial\mathbf{E}}{\partial t}=\frac{1}{\kappa\mu\Pi}\left(\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}\right)=a^2\left(\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}\right);\quad a^2=\frac{1}{\kappa\mu\Pi}. \tag{12}$$

3. Physical Meaning of the Complex Impulse Representation.

The basic equations pertaining to temperature fields and eddy-current fields being linear ones, the actions of the individual external forces may be superimposed to the total field. It appears natural to interpret physically the complex impulse integrals by introduction of the conception of a *Schwingungspaket.* The impulse should be interpreted as the superposition of an infinite number of elementary forces, continuously (i.e. for all time values) oscillating with the complex angular velocity p. The magnitude of the quantity under the integral sign is called in this conception the *frequency-spectrum* of the impulse.

By means of the above interpretation of the complex impulse-integrals the investigation of transient phenomena is formally reduced to the knowledge of the forced oscillations with the complex angular velocity p, and the whole transient process is obtained by integration extended over infinitely many elementary oscillations of the complex angular velocity p. The partial differential equations of the field then simplify to

$$\Delta\vartheta = \frac{p}{\ }\vartheta \tag{13}$$

and

$$\Delta\mathbf{E} = \ ;\mathbf{E}, \tag{14}$$

where Δ stands for the differential operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Let us carry out the integration and the functional-theoretic treatment of these equations in the following examples.

4. Machine Model for Heating Theory.

To assure a possibly clear conception of the nature of heating phenomena in electric machinery let us consider a simplified model as indicated in Fig. 90. Let the stator and rotor be heat-insulated from each other by means of the air gap. If we neglect the curvature of the armature surface in order to simplify matters, the stator or rotor of the machine (for heating calculations) can be replaced by a core stacking carrying the winding. It is there that in addition to the winding losses the main part of the core losses is created, so that we may regard

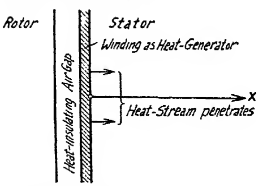

Fig. 90. Simplest machine model of the heating phenomena.

the bounding surface of the core stacking as the seat of the heat sources. The direction of the heat flow in the core is vertical to the surface.

Let S be the heat energy per unit area of the core. Assuming the x-coördinate as starting at the surface and going into the inside of the core, the equation of heat conduction for the complex angular velocity p becomes:

$$\frac{d^2\vartheta}{dx^2}=\frac{p}{a^2}\vartheta. \tag{15}$$

Let us integrate this equation for different boundary conditions and deduce the resulting laws of heating, using this integral in connection with the complex impulse integrals. To this purpose we split up S into elementary parts dS, each of which possesses the complex frequency p; the splitting-up is performed by means of some of the previously mentioned impulse formulas.

5. The Heating Curve for Pure Conduction Heat Rejection.

We first deal with a machine having an infinitely thick core stacking, the heat rejection thus taking its course solely through the core. The corresponding solution of equation (15) is:

$$\vartheta=Ae^{-\frac{\sqrt{p}}{a}x} \text{ with Re } (\sqrt{p})>0. \tag{16}$$

The integration constant A follows from the boundary condition

$$w_0=\left(-\lambda\frac{d\vartheta}{dx}\right)_{x=0}=A\cdot\lambda\frac{\sqrt{p}}{a}=dS;\quad A=\frac{dS\cdot a}{\sqrt{p}\cdot\lambda}. \tag{17}$$

If the heating is brought about by a constant load S starting at the time $t=0$, we have

$$S(t)=\frac{S}{2\pi i}\cdot\int_{-i\infty}^{+i\infty}\frac{e^{pt}dp}{p}. \tag{18}$$

Hence we obtain for the heating curve the complex integral:

$$\vartheta=\frac{a}{\lambda}\cdot S\cdot\frac{1}{2\pi i}\cdot\int_{-i\infty}^{+i\infty}\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}x}{a}}dp}{p\sqrt{p}}=\frac{a}{\lambda}\cdot S\cdot g(t), \tag{19}$$

if we put

$$g(t)=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}x}{a}}dp}{p\sqrt{p}}. \tag{19a}$$

To evaluate the integral we first form

$$g'(t)=\frac{1}{2\pi i}\int^{+i\infty}\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}}{a}x}}{\sqrt{p}}\,dp \tag{20}$$

The integral in equation (20) is the integral of a *many-valued function.* In order to make use of Cauchy's integral theorem for the computation of this integral we must restrict ourselves to *one* sheet of the Riemann surface. Here, we must choose that sheet on which the real part of $\sqrt{p}$ is positive, because otherwise the temperature does not vanish for $x\to\infty$ (compare equation (16)). The two sheets of our Riemann surface are connected together at the "branch-cut" on which Re $(\sqrt{p})$ vanishes, i.e. p is negative; hence, the negative real axis should be chosen as a branch-cut (Fig. 91). Introducing polar coördinates*

$$p=\rho e^{i\vartheta} \tag{21}$$

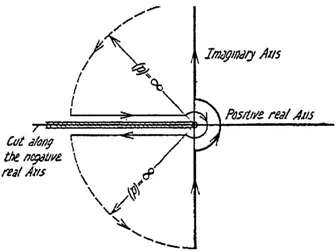

Fig. 91. Transposition of the original path of integration to the branch-cut of the p-plane.

we have on the bottom side of the branch-cut

$$p=\sqrt{\rho}\cdot e^{-i\frac{\pi}{2}}=-i\sqrt{\rho} \tag{22}$$

and on the top side

$$p=\sqrt{\rho}\cdot e^{+i\frac{\pi}{2}}=+i\sqrt{\rho} \tag{22a}$$

We now apply Cauchy's integral theorem to the closed contour consisting of the imaginary axis, the semi-circle in Re $(p)<0$ about the origin and the branch-cut; note, that we now have to exclude the origin by means of a complete circle $|p|=\rho_0$. Inside of the traversed region the function under the integral sign is one-valued and regular. Therefore, we have in self-explanatory symbolic notation:

$$\frac{1}{2\pi i}\int\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}}{a}x}}{\sqrt{p}}\,dp+\frac{1}{2\pi i}\int\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}}{a}x}}{\sqrt{p}}\,dp=0, \tag{23}$$

because the infinitely large quarter-circles in $\mathrm{Re}(p)<0$ do not contribute anything to the integral. By equation (23), we may transfer the integral from the original path of integration to the branch-cut, thus obtaining

$$g'(t)=\frac{1}{2\pi i}\int\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}}{a}x}}{\sqrt{p}}\,dp \tag{23a}$$

* Here and in the following discussion let ϑ temporarily designate the arcus of the complex quantity p.

Corresponding to the indicated path of integration, the total integral splits-up into the following three component integrals:

$$I_1=\int_{-\infty}^{-\rho_0}\frac{e^{pt}\cdot e^{+i\sqrt{|p|}\frac{x}{a}}}{-i\sqrt{|p|}}dp=\int_{\rho_0}^{\infty}\frac{e^{-pt}\cdot e^{i\sqrt{p}\frac{x}{a}}}{-i\sqrt{p}}\,dp,$$

$$I_2=\int_{-\pi}^{+\pi}\frac{e^{\rho_0 e^{i\vartheta}\cdot t}\cdot e^{-\frac{\sqrt{\rho_0}}{a}\cdot e^{i\frac{\vartheta}{2}}\cdot x}\cdot\rho_0\, e^{i\vartheta}d\vartheta}{\sqrt{\rho_0}\cdot e^{i\frac{\vartheta}{2}}}, \tag{24}$$

$$I_3=\int_{-\rho_0}^{-\infty}\frac{e^{pt}\cdot e^{-i\sqrt{|p|}\frac{x}{a}}\,dp}{i\sqrt{|p|}}=-\int_{\rho_0}^{\infty}\frac{e^{-pt}\cdot e^{-i\sqrt{p}\frac{x}{a}}}{i\sqrt{p}}\,dp.$$

On the first integral we use the identity

$$pt-i\sqrt{p}\,\frac{x}{a}\equiv\left(\sqrt{pt}-\frac{i}{2}\frac{x}{a\sqrt{t}}\right)^2+\frac{x^2}{4a^2t}$$

and then substitute

$$\left(\sqrt{pt}-\frac{i}{2}\frac{x}{a\sqrt{t}}\right)=-u,$$

so that

$$I_1=\int^{\infty}\frac{e^{-pt}\cdot e^{i\sqrt{p}\frac{x}{a}}\,dp}{-i\sqrt{\ }}=-\frac{e^{-\frac{x^2}{4a^2t}}\cdot 2i}{\sqrt{t}}\cdot\int_{-\sqrt{\rho_0 t}+\frac{i}{2}\frac{x}{a\sqrt{t}}}^{-\infty+\frac{i}{2}\frac{x}{a\sqrt{t}}}e^{-u^2}du. \tag{24a}$$

On the third integral, because of

$$pt+i\sqrt{p}\,\frac{x}{a}\equiv\left(\sqrt{pt}+\frac{i}{2}\frac{x}{a\sqrt{t}}\right)^2+\frac{x^2}{4a^2t},$$

and using the substitution

$$\left(\sqrt{pt}+\frac{i}{2}\frac{x}{a\sqrt{t}}\right)=+u$$

we have the transformation:

$$I_3=-\int_{\rho_0}\frac{e^{-pt}\cdot e^{-i\sqrt{p}\frac{x}{a}}\,dp}{i\sqrt{p}}=\frac{e^{-\frac{x^2}{4a^2t}}\cdot 2i}{\sqrt{t}}\cdot\int_{\sqrt{\rho_0 t}+\frac{i}{2}\frac{x}{a\sqrt{t}}}^{\infty+\frac{i}{2}\frac{x}{a\sqrt{t}}}e^{-u^2}\,du \tag{24b}$$

Integral I_2 vanishes at the limit $\rho_0 \to 0$ as it is proportional to $\sqrt{\rho_0}$. Consequently, we have

$$q'(t) = \frac{1}{2\pi i} \cdot \lim_{\rho_0 \to 0} (I_1 + I_3) = \frac{1}{\pi} \cdot \int_{-\infty + \frac{i}{2}\frac{x}{a\sqrt{t}}}^{+\infty + \frac{i}{2}\frac{x}{a\sqrt{t}}} e^{-u^2} du \cdot \frac{e^{-\frac{x^2}{4a^2t}}}{\sqrt{t}} \cdot \tag{25}$$

This integral may be transferred to the real axis by Cauchy's integral theorem and then results in the value $\sqrt{\pi}$ of the error-integral. We thus obtain ultimately

$$g'(t) = \frac{1}{\sqrt{\pi t}} \cdot e^{-\frac{x^2}{4a^2t}} \equiv \frac{1}{\sqrt{\pi \frac{4a^2t}{x^2}}} \cdot e^{-\frac{x^2}{4a^2t}} \cdot \frac{2a}{x} \tag{25a}$$

and for the temperature, by equation (19) and (20), compare Fig. 92:

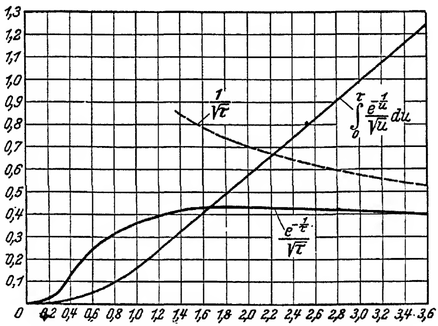

Fig. 92. Heating curve for pure conduction heat-rejection.

$$= \frac{a}{\lambda} \cdot S \cdot \int_0 \frac{1}{\sqrt{\pi t}} \cdot e^{-\frac{x^2}{4a^2t}} dt \equiv \frac{2a^2}{\lambda x} \cdot S \cdot \int_0 \frac{1}{\sqrt{\pi \frac{4a^2t}{x^2}}} \cdot e^{-\frac{x^2}{4a^2t}} dt$$

$$= \frac{x}{2\lambda} \cdot S \int^{\tau} \frac{1}{\sqrt{\pi\tau}} \cdot e^{-\frac{1}{\tau}} d\tau, \tag{25b}$$

where $\tau = \frac{4a^2t}{x^2}$ denotes the numeric "time of locus.'

We are most interested in the temperature of the winding ($x=0$):

$$\vartheta_0 = \frac{a}{\lambda} \cdot S \cdot \int_0^t \frac{dt}{\sqrt{\pi t}} = \frac{a}{\lambda} \cdot S \cdot \frac{2}{\sqrt{\pi}} \cdot \sqrt{t}. \tag{26}$$

Under the assumed cooling conditions the winding thus does not reach a steady state, but, finally, will have to burn out; the same conclusion holds also for the core, as we can readily see from equation (25b) and from Fig. 92 that the temperature rises with the square root of t, i.e. unlimitedly.

6. The Heating Curve for External Cooling.

The unlimited temperature rise found in the preceding article can be stopped by external cooling. Let us assume that the core stacking is of the finite width d (Fig. 93); let, at the outside boundary surface, a cooling-off take place according to Newton's law of cooling

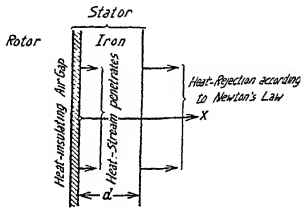

Fig. 93. Machine model for external cooling.

$$-\lambda \left(\frac{d\vartheta}{dx}\right)_{x=} = h \cdot \vartheta, \tag{27}$$

where h is the coefficient of external heat conductivity. For the course of the temperature with the complex frequency p we have

$$\vartheta = A \cdot e^{-\frac{\sqrt{p}}{a}x} + Be^{+\frac{\sqrt{p}}{a}x} \tag{28}$$

and, thus, from the boundary conditions, equation (17) and (27):

$$A = \frac{dS \cdot a}{\sqrt{p} \cdot \lambda} \cdot \frac{1 + \dfrac{ha}{\sqrt{p\lambda}}}{2 \sinh \dfrac{\sqrt{p}d}{a} + \dfrac{ha}{\sqrt{p\lambda}} \cdot 2 \cosh \dfrac{\sqrt{p}d}{a}} \cdot e^{\sqrt{p}d}$$

$$B = \frac{dS \cdot a}{\sqrt{p} \cdot \lambda} \cdot \frac{1 - \dfrac{ha}{\sqrt{p\lambda}}}{2 \sinh \dfrac{\sqrt{p}d}{a} + \dfrac{ha}{\sqrt{p\lambda}} \cdot 2 \cosh \dfrac{\sqrt{p}d}{a}} \cdot {}^{\sqrt{p}d} \tag{29}$$

For a suddenly applied load the heating curve is hence obtained from the complex integral

$$\vartheta = \mathrm{S}\,\frac{a}{\lambda}\cdot\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{e^{pt}dp}{p\sqrt{p}}\cdot\frac{\cosh\frac{\sqrt{p}(d-x)}{a}+\frac{ha}{\sqrt{p}\lambda}\cdot\sinh\frac{\sqrt{p}(d-x)}{a}}{\sinh\frac{\sqrt{p}d}{a}+\frac{ha}{\sqrt{p}\lambda}\cdot\cosh\frac{\sqrt{p}d}{a}}. \tag{30}$$

Although the function under the integral sign here too shows $\sqrt{p}$, it is obviously a *one-valued function* of p and, therefore, has to be represented on the plain p-plane; we obtain the temperature explicitly by integrating over the poles. This is, in principle, the contents of Heaviside's theorem (compare Part I, Sect. F, Art. 9). The poles are the zeros of the denominator:

$$p=0 \text{ and } \frac{ha}{\sqrt{p}\lambda}+\tanh\frac{\sqrt{p}\lambda}{a}=0. \tag{31}$$

The pole $p=0$ furnishes the steady state temperature:

$$\vartheta_\infty = \mathrm{S}\,\frac{a}{\lambda}\cdot\frac{1+\frac{h}{\lambda}(d-x)}{\frac{ha}{\lambda}} = \frac{\mathrm{S}}{h}\left[1+\frac{h}{\lambda}(d-x)\right], \tag{32}$$

where the first term tells us about the temperature rise at the boundary $x=d$, the second — the temperature drop in the core. All the rest of the poles are negative; we, therefore, may put $\sqrt{p}=iq$ and obtain for q the real transcendental equation

$$\frac{ha}{q\lambda}\equiv\left(\frac{a}{qd}\right)\cdot\frac{hd}{\lambda}=\tan\frac{qd}{a} \tag{33}$$

or, with $\frac{qd}{a}=\nu$,

$$\frac{hd}{\lambda}\cdot\frac{1}{\nu}=\tan\nu;\quad \text{with the roots: } p_\nu=-q_\nu^2=-\frac{a^2}{d^2}\nu^2. \tag{33a}$$

From equation (30) the heating curve follows:

$$\frac{\vartheta}{\vartheta_\infty}=1+\frac{\frac{ha}{\lambda}}{1+\frac{h}{\lambda}(d-x)}\sum\frac{e^{-\frac{a^2}{d^2}\nu^2 t}}{p_\nu\sqrt{p_\nu}}\cdot\frac{\cos\nu\frac{d-x}{d}+\frac{hd}{\lambda\nu}\cdot\sin\nu\frac{d-x}{d}}{\frac{\partial}{\partial p}\left(\sinh\frac{\sqrt{p}d}{a}+\frac{ha}{\sqrt{p}\lambda}\cdot\cosh\frac{\sqrt{p}d}{a}\right)_{p=p_\nu}}. \tag{34}$$

But,

$$\frac{\partial}{\partial p}\left(\sinh\frac{\sqrt{p}d}{a}+\frac{ha}{\sqrt{p}\lambda}\cdot\cosh\frac{\sqrt{p}d}{a}\right)$$

$$=\left(\frac{d}{a}\cosh\frac{\sqrt{p}d}{a}-\frac{ha}{\sqrt{p}\lambda}\cdot\cosh\frac{\sqrt{p}d}{a}+\frac{hd}{\sqrt{p}\lambda}\cdot\sinh\frac{\sqrt{p}d}{a}\right)\cdot\frac{1}{2}\frac{1}{\sqrt{p}},$$

and thus:

$$p_\nu\sqrt{p_\nu}\cdot\frac{\partial}{\partial p}\left(\sinh\frac{\sqrt{p}d}{a}+\frac{ha}{\sqrt{p}\lambda}\cosh\frac{\sqrt{p}d}{a}\right)$$

$$=-\frac{a}{d}\cdot\frac{1}{2}\nu^2\cdot\left\{\left(1-\frac{ha^2}{p_\nu d\lambda}\cosh\frac{\sqrt{p_\nu}d}{a}\right)+\frac{ha}{\sqrt{p_\nu}\lambda}\cdot\sinh\frac{\sqrt{p_\nu}d}{a}\right. \tag{34a}$$

$$=-\frac{a}{d}\cdot\frac{1}{2}\nu^2\cdot\left\{\left(1+\frac{hd}{\nu^2\lambda}\right)\cdot\cos\nu+\frac{hd}{\lambda\nu}\cdot\sin\nu\right.$$

With $\zeta=\dfrac{x}{a}$ the heating curve therefore becomes:

$$\frac{\vartheta}{\vartheta_\infty}=1-\frac{h}{\lambda+h(1+\zeta)}\cdot\sum_\nu\frac{e^{-\frac{a^2}{d^2}\nu^2 t}}{\nu^2}\cdot\frac{\cos\nu(1-\zeta)+\frac{hd}{\lambda\nu}\cdot\sin\nu(1-\zeta)}{\left(1+\frac{hd}{\lambda\nu^2}\right)\cos\nu+\frac{hd}{\lambda\nu}\sin\nu}. \tag{34b}$$

This formula simplified considerably for the limiting case $h\longrightarrow\infty$, which case corresponds to a temperature $\vartheta=0$ kept constant at the cooled end; for then $\nu=\dfrac{\pi}{2},\dfrac{3\pi}{2},\dfrac{5\pi}{2},\ldots$, and hence:

$$\frac{\vartheta}{\vartheta_\infty}=1-\frac{8}{\pi^2}\cdot\frac{1}{1-\zeta}\cdot\sum_{k=1,3,5\ldots}\frac{e^{-\frac{\pi^2}{4}k^2\frac{a^2}{d^2}t}}{k^2}\cdot\frac{\sin k\frac{\pi}{2}(1-\zeta)}{\sin k\frac{\pi}{2}}. \tag{35}$$

Finally, putting $\dfrac{\pi^2}{4}\cdot\dfrac{a^2}{d^2}t=\dfrac{t}{T}=\tau$, where $T=\dfrac{4}{\pi^2}\cdot\dfrac{d^2}{a^2}$ is the time constant of the process, we obtain:

$$\frac{\vartheta}{\vartheta_\infty}=1-\frac{8}{\pi^2}\cdot\frac{1}{1-\zeta}\cdot\sum_{k=1,3,5\ldots}\frac{e^{-k^2\tau}}{k^2}\cdot\frac{\sin k\frac{\pi}{2}(1-\zeta)}{\sin k\frac{\pi}{2}}. \tag{35a}$$

The heating curve of the winding we get for $\zeta = 0$:

$$\frac{\vartheta}{\vartheta_\infty} = 1 - \frac{8}{\pi^2} \cdot \left[\frac{e^{-\tau}}{1} + \frac{e^{-9\tau}}{9} + \frac{e^{-25\tau}}{25} + \cdots \right], \qquad (35b)$$

so that at the beginning of the process the temperature rises more rapidly than according to the exponential law. In the neighborhood of the cooled end, however, where $(1-\zeta)$ is small, we have:

$$\frac{\vartheta}{\vartheta_\infty} = 1 - \frac{4}{\pi^2} \cdot \left[\frac{e^{-\tau}}{1} - \frac{e^{-9\tau}}{3} + \frac{e^{-25\tau}}{5} - + \cdots \right] = 1 - \frac{4}{\pi^2} \tan^{-1} e^{-\tau}. \quad (35c)$$

Here, the temperature at the beginning rises slower than according to the exponential law (Figs. 94 and 95). An approximately exponential temperature course is encountered at the distance ζ_0 at which the second term of the series in equation (35a) vanishes; this happens when

$$3\frac{\pi}{2}(1-\zeta) = \pi$$

and thus $(1-\zeta) = \frac{2}{3}$ or $\zeta = \frac{1}{3}$.

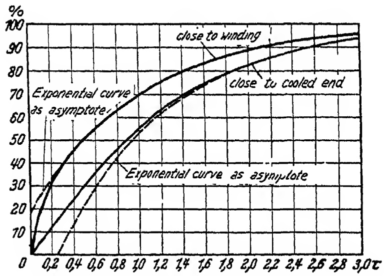

Fig. 94. Heating curves of the machine with external cooling.

7. The Heating Curve for Internal Cooling.

We now deal with the heating of a machine being cooled directly at the winding by an air current. Again we write down Newton's law of cooling. Let us assume the core stacking to be of infinite width, so that it acts only at the beginning as a heat accumulator. The general solution is again:

$$\vartheta = A \cdot e^{-\frac{\sqrt{p}\,x}{a}}.$$

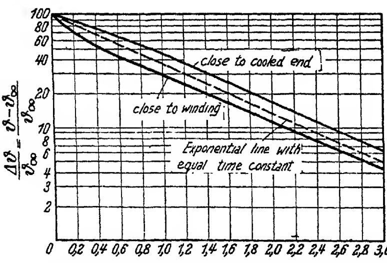

Fig. 95. Logarithmic representation of the law of heating for external cooling.

The boundary conditions are: the inside conduction heat-current into the core and the heat-current into the air gap are connected in parallel, thus:

$$dS=\lambda\frac{\sqrt{p}}{a}\cdot A+hA=\lambda\left[\frac{\sqrt{p}}{a}+h'\right]A, \qquad \text{where } h'=\frac{h}{\lambda}. \tag{36}$$
$$A=dS\cdot\frac{a}{\lambda}\cdot\frac{}{h'a+\sqrt{p}}.$$

We, therefore, win the heating curve in the form of the following integral:

$$\vartheta=S\cdot\frac{a}{\lambda}\cdot\frac{1}{2\pi i}\int^{+i\infty}\frac{e^{pt}\cdot\quad dp}{p(h'a+\sqrt{p})}\equiv S\cdot\frac{a}{\lambda}\cdot f(t). \tag{37}$$

The function under the integral sign, i.e. the function for the calculation of $f(t)$, is a many-valued function, which may be transformed into a one-valued function by means of the already previously used branch-cut of the p-plane along the negative real axis; we also transfer the integral to this branch-cut and obtain:

$$f(t)=\frac{1}{2\pi i}\cdot\int\frac{e^{pt}\cdot e^{-\frac{\sqrt{p}}{a}x}\,dp}{p(h'a+\sqrt{p})}=\frac{1}{2\pi i}\cdot I. \tag{37a}$$

Again we have to split-up the integral I into three components I_1, I_2 and I_3:

$$\begin{aligned}I=I_1+I_2+I_3=&\int^{-\rho_0}\frac{e^{pt}\cdot e^{i\frac{\sqrt{|p|}x}{a}}dp}{p(h'a-i\sqrt{|p|})}\\&+\int_{-\pi}^{+\pi}\frac{e^{\rho_0e^{i\vartheta}\cdot t}\cdot e^{-\sqrt{\rho_0}\cdot e^{i\frac{\vartheta}{2}}\cdot\frac{x}{a}}d(\rho_0e^{i}}{\rho_0e^{i\vartheta}\cdot\left(h'a+\sqrt{\rho_0}\cdot e^{i\frac{\vartheta}{2}}\right)}\\&+\int^{\infty}\frac{e^{pt}\cdot e^{-i\frac{\sqrt{|p|}x}{a}}dp}{p(h'a+i\sqrt{|p|})}\end{aligned} \tag{37b}$$

Substitute $p=-u$ in I_1, finding

$$I_1=\int_{\infty}^{\rho_0}\frac{e^{-ut}\cdot e^{i\sqrt{u}\frac{x}{a}}}{u(h'a-i\sqrt{u})}\,du=-\int_{\rho_0}\frac{e^{-ut}\cdot e^{i\sqrt{u}\frac{x}{a}}}{u(h'a-i\sqrt{u})}\,du.$$

At the limit $\rho_0\to0$ and $t\to\infty$ the neighborhood of the zero furnishes the main contribution to the integral, as the function under the integral sign,

under the influence of the exponential term, rapidly becomes very small for increasing u. We, therefore, may develop the denominator into a series in powers of $\sqrt{u}$, thus obtaining the following representation (this is the so-called "saddle-point" procedure):

$$I_1 = -\int^{\infty} \frac{e^{-ut} \cdot e^{i\sqrt{u}\frac{x}{a}}}{u \cdot h'a} du \left(1 + i\frac{\sqrt{u}}{h'a} + i^2\left(\frac{\sqrt{u}}{h'a}\right)^2 + \cdots\right).$$

In like manner I_3 may be expressed by the series:

$$I_3 = \int_{\rho_0}^{\infty} \frac{e^{-ut} \cdot e^{i\sqrt{u}\frac{x}{a}} du}{u(h'a + i\sqrt{u})} = \int_{\rho_0}^{\infty} \frac{e^{-ut} \cdot e^{-i\sqrt{u}\frac{x}{a}} du}{uh'a} \left(1 - i\frac{\sqrt{u}}{h'a} + i^2\left(\frac{\sqrt{u}}{h'a}\right)^2 - \cdots\right).$$

In abbreviated notation we, therefore, have:

$$I_1 + I_3 = -2i \int_{\rho_0} \frac{e^{-ut} \cdot \sin \frac{\sqrt{u}x}{a}\, du}{uh'a} \cdot \left(1 + \frac{\frac{d}{dt}}{(h'a)^2} + \frac{\frac{d^2}{dt^2}}{(h'a)^4} + \cdot\right.$$

$$\left. -2i \int_{\rho_0}^{\infty} \frac{e^{-ut} du}{uh'a} \frac{\sqrt{u}}{h'a} \cos \frac{\sqrt{ux}}{a} \cdot \left(1 + \frac{\frac{d}{dt}}{(h'a)^2} + \frac{\frac{d^2}{dt^2}}{(h'a)^4} + \cdots\right)\right.$$

Now, consider the integral

$$I = 2\int_0^{\infty} \frac{e^{-ut}}{\sqrt{u}} \cdot \cos \frac{\sqrt{u}x}{a} \cdot du \equiv \int_0^{\infty} \frac{e^{-ut}}{\sqrt{u}} \cdot e^{i\sqrt{u}\frac{x}{a}} du + \int_0^{\infty} \frac{e^{-ut}}{\sqrt{u}} \cdot e^{-i\sqrt{u}\frac{x}{a}} du.$$

Substituting $\left(\sqrt{ut} + \frac{i}{2}\frac{x}{a\sqrt{t}}\right) = v$ in the first integral and $\left(\sqrt{ut} - \frac{i}{2}\frac{x}{a\sqrt{t}}\right) = -v$ in the second one [compare also equations (23a) and (24)], we have:

$$\frac{4x^2t}{\sqrt{t}} \cdot \quad 2\int_{\frac{i}{2}\frac{x}{a\sqrt{t}}} e^{-v^2} dv - 2\int_{\frac{i}{2}\frac{x}{a\sqrt{t}}} e^{-v^2} dv \qquad \frac{e^{-\frac{x^2}{4a^2t}}}{\sqrt{t}} \cdot 2\sqrt{\pi}.$$

Furthermore,

$$\left(\frac{x}{a}\right) \Big\{ \int_0 \frac{e^{-ut}}{u} \sin \sqrt{u}\frac{x}{a} \cdot du \Big\} = \frac{1}{2} I$$

and, therefore, with $v=\frac{\zeta}{2\sqrt{t}}$:

$$\int_0^\infty \frac{e^{-ut}}{u}\sin\sqrt{u}\,\frac{x}{a}\cdot du=\int_0^\infty \frac{e^{-\frac{\zeta^2}{4t}}\sqrt{\pi}}{\sqrt{t}}\,d\zeta=2\sqrt{\pi}\int_0^{2a\sqrt{t}} e^{-v^2}dv=\pi\cdot\Phi\left(\frac{x}{a\sqrt{t}}\right),$$

where Φ denotes Gauss' error function.*

For $\rho_0\rightarrow 0$ we hence obtain:

$$I_1+I_3=\left(1+\frac{\frac{d}{dt}}{(h'a)^2}+\frac{\frac{d^2}{dt^2}}{(h'a)^4}+\cdots\right)\cdot\left[-\frac{2\pi i}{(h'a)^2}\cdot\frac{e^{-\frac{x^2}{4at}}}{\sqrt{\pi t}}-\frac{2\pi i}{h'a}\cdot\Phi\left(\frac{x}{2a\sqrt{t}}\right)\right].$$

Integral I_2 may be calculated as residue for $\rho_0\rightarrow 0$:

$$\lim_{\rho_0\rightarrow 0} I_2=\frac{2\pi i}{h'a}.$$

The complete function then is:

$$f(t)=\frac{1}{h'a}-\left(1+\frac{\frac{d}{dt}}{(h'a)^2}+\frac{\frac{d^2}{dt^2}}{(h'a)^4}+\cdots\right)\left[\frac{e^{-\frac{x^2}{4a^2t}}}{(h'a)^2\cdot\sqrt{\pi\cdot t}}+\frac{\Phi\left(\frac{x}{2a\sqrt{t}}\right)}{h'a}\right]. \quad (38)$$

The first part represents the steady state temperature ϑ_∞, and the heating curve thus is:

$$\frac{\vartheta}{\vartheta_\infty}=1-\left(1+\frac{\frac{d}{dt}}{(h'a)^2}+\frac{\frac{d^2}{dt^2}}{(h'a)^4}+\cdots\right)\left[\frac{e^{-\frac{x^2}{4a^2t}}}{h'a\sqrt{\pi t}}+\Phi\left(\frac{x}{2a\sqrt{t}}\right)\right]. \quad (38a)$$

While in the first example the integration had to be carried out along a *branch-cut* and in the second example the integration was restricted to calculations of *residues*, we had to use here both typical integration procedures.

The series in equation (38a) simplifies considerably for $x=0$, so that we find for the winding temperature (Fig. 96):

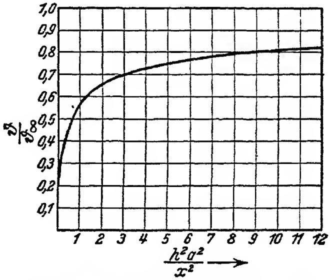

Fig. 96. Heating curve of a winding of an internally cooled machine.

* Compare: Jahnke-Emde, "Funktionentafeln," p. 31.

$$\frac{\vartheta}{\vartheta_\infty}=1-\left(1+\frac{\frac{d}{dt}}{(h'a)^2}+\frac{\frac{d^2}{dt^2}}{(h'a)^4}+\cdots\right)\cdot\frac{1}{h'a\cdot\sqrt{\pi t}}$$

$$=1-\frac{1}{h'a\cdot\sqrt{\pi t}}\cdot\left(1-\frac{1}{2}\frac{1}{(h'a)^2}\cdot\frac{1}{t}+\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{(h'a)^4}\cdot\frac{1}{t^2}-+\cdots\right). \quad (38b)$$

The summation of the above series may be performed,* the series yielding:

$$\frac{\vartheta}{\vartheta_\infty}=1-e^{(h'a)^2t}\cdot[1-\Phi(h'a\sqrt{t})]=1-e^{-\frac{h^2a^2}{\lambda^2}t}\cdot\left[1-\Phi\left(\frac{ha}{\lambda}\sqrt{t}\right)\right].$$

For very small h-values we obtain from the above

$$\lim_{h\to\infty}\vartheta=\mathrm{S}\cdot\frac{a}{\lambda}\cdot\frac{\lambda}{ha}\left[1-1+\frac{2}{\sqrt{\pi}}\cdot\frac{ha}{\lambda}\sqrt{t}\right]=\mathrm{S}\cdot\frac{a}{\lambda}\cdot\frac{2}{\sqrt{\pi}}\cdot\sqrt{t}, \quad (38c)$$

in agreement with equation (26).

8. The Switching-in of a Machine with Eddy-Current Rotor.

Let us apply the complex treatment of transient phenomena to the calculation of the stator current impulse of a rotating field eddy-current rotor machine. Let τ be the pole pitch and δ the air gap of such a machine. As no eddy-currents flow in the air gap, we have for the eddy field $\mathbf{E}$ of the air gap the equation

$$\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}=0. \quad (39)$$

As for the equation of the rotor field, we have to consider the relative velocity of the moving masses and the rotating field as observed from the stator. We succeed in doing this by generalizing the conception of the slip as follows:

$$s=\frac{p-i\omega_m}{p} \quad (40)$$

where ω_m is the (real) angular velocity of the mechanic rotational motion. We thus obtain for the rotor field (compare Art. 2):

$$\left.\begin{aligned}\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}=k^2\mathbf{E};\quad k^2&=4\pi\kappa\mu p\cdot\frac{p-i\omega_m}{p}\cdot 10^{-9}\\&=4\pi\kappa\mu(p-i\omega_m)\cdot 10^{-9}\end{aligned}\right\} \quad (41)$$

To simplify matters, let us assume the rotor winding to be a homogeneous layer of thickness Δ and of permeability $\mu=1$ carrying the eddy-current

* See: Jahnke-Emde, "Funktionentafeln," p. 31.

(Fig. 97). Equations (39) and (40) may then be integrated without difficulty for fields varying periodically in x with the pole pitch τ. Neglecting the ohmic stator resistance and the magnetic resistance of the core, we arrive at the stator current A as a function of the electric field strength $\mathbf{E}_0$ of the stator periphery. The resulting current equation is of the form*:

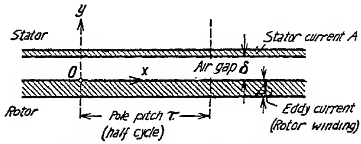

Fig. 97. Machine model of an eddy current rotor.

$$A = \mathbf{E}_0 \cdot \frac{1}{4\tau \cdot 10^{-9} \cdot p} \cdot \tanh \frac{\pi\delta}{\tau} \frac{1+\sqrt{1+\frac{\tau^2}{\pi^2}k^2} \coth \frac{\pi}{\tau}\delta \cdot \tanh\left(\frac{\pi}{\tau}\Delta\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\right)}{1+\sqrt{1+\frac{\tau^2}{\pi^2}k^2} \tanh \frac{\pi}{\tau}\delta \cdot \tanh\left(\frac{\pi}{\tau}\Delta\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\right)}. \tag{42}$$

The above relation, if in particular we put $p = i\omega$, tells us all about the steady state behavior of an induction type rotating field eddy-current rotor machine.

Let now the transient phenomenon be produced by suddenly (at $t=0$) impressing the voltage of a three-phase system upon the terminals of our machine running at synchronous speed ($\omega_m = \omega$). A steady state electric rotating field being represented by

$$\mathbf{E} = \mathbf{E}_0 \cdot e^{i\left(\omega t - \frac{\pi}{\tau}x\right)}, \tag{43}$$

we find the complex representation of the suddenly switched-in rotating field to be the hook integral:

$$\mathbf{E} = \mathbf{E}_0 \cdot \frac{1}{2\pi i} \cdot \int_{-i\infty}^{+i\infty} \frac{e^{pt} \cdot e^{-i\frac{\pi}{\tau}x}\, dp}{p - i\omega}. \tag{43a}$$

As the distribution of the field in space is known to begin with and thus is non-essential for this problem, we obtain from the above, respecting equation (42), the current equation in the form:

$$A = \mathbf{E}_0 \cdot \frac{1}{4\tau \cdot 10^{-9}} \cdot \tanh \frac{\pi\delta}{\tau} \cdot \int_{-i\infty}^{+i\infty} \frac{e^{pt} \cdot dp}{p(p-i\omega)} \cdot$$

$$\frac{1+\sqrt{1+\frac{\tau^2}{\pi^2}k^2} \cdot \coth \frac{\pi\delta}{\tau} \cdot \tanh\left(\frac{\pi\Delta}{\tau} \cdot \sqrt{1+\frac{\tau^2}{\pi^2} \cdot k^2}\right)}{1+\sqrt{1+\frac{\tau^2}{\pi^2}k^2} \cdot \tanh \frac{\pi\delta}{\tau} \cdot \tanh\left(\frac{\pi\Delta}{\tau} \cdot \sqrt{1+\frac{\tau^2}{\pi^2} \cdot k^2}\right)}. \tag{44}$$

* See F. Ollendorff: *Arch. Elektrot.*, vol. 24 (1930), p. 129 ff.; note in particular page 144.

As we can easily convince ourselves of the fact that the function under the integral sign is a one-valued function, the integral may be found by summation over all poles, according to the rules of residue calculus. The pole

$$p=0 \tag{45}$$

furnishes the direct current component of the impulse current:

$$A_{d.c.}=\mathrm{E}_0\cdot\frac{1}{4\tau\cdot 10^{-9}}\cdot\tanh\frac{\pi\delta}{\tau}\,\frac{1}{(-i\omega)}\,\frac{1+\sqrt{1+\frac{\tau^2}{\pi^2}k_\omega^2}\cdot\coth\frac{\pi\delta}{\tau}\cdot\tanh\frac{\pi\Delta}{\tau}\cdot\sqrt{1+\frac{\tau^2}{\pi^2}k_\omega^2}}{1+\sqrt{1+\frac{\tau^2}{\pi^2}k_\omega^2}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\tanh\frac{\pi\Delta}{\tau}\cdot\sqrt{1+\frac{\tau^2}{\pi^2}k_\omega^2}}, \tag{46}$$

or, simplified:

$$\cong -\mathrm{E}_0\cdot\frac{1}{4\tau\cdot 10^{-9}\cdot i\omega}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\frac{1+\frac{\tau}{\pi}\cdot k_\omega\cdot\coth\frac{\pi\delta}{\tau}\cdot\tanh\Delta k_\omega}{1+k_\omega\cdot\tanh\frac{\pi\delta}{\tau}\cdot\frac{\tau}{\pi}\cdot\tanh\Delta k_\omega}. \tag{47}$$

The pole

$$p=i\omega$$

indicates the sustained short-circuit current:

$$A_s=\mathrm{E}_0\cdot\frac{1}{4\tau\cdot 10^{-9}}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\frac{e^{i\omega t}}{i\omega}\cdot\frac{1+\coth\frac{\pi\delta}{\tau}\cdot\tanh\frac{\pi\Delta}{\tau}}{1+\tanh\frac{\pi\delta}{\tau}\cdot\tanh\frac{\pi\Delta}{\tau}}. \tag{48}$$

The remaining poles result from the transcendental equation

$$\frac{\frac{\pi\Delta}{\tau}\cdot\coth\frac{\pi\delta}{\tau}}{\frac{\pi\Delta}{\tau}\cdot\sqrt{1+\frac{\tau^2}{\pi^2}k^2}}=-\tanh\frac{\pi\Delta}{\tau}\sqrt{1+\frac{\tau^2}{\pi^2}k^2} \tag{49}$$

or, with $\frac{\pi\Delta}{\tau}\cdot\sqrt{1+\frac{\tau^2}{\pi^2}k^2}=i\nu$, from $\frac{\pi\Delta}{\tau}\cdot\coth\frac{\pi\delta}{\tau}=\nu\cdot\tan\nu$. (49a)

They are decisive for the alternating current component of the impulse

short-circuit current. Fig. 98 indicates these roots of the equation: we recognize that they rapidly approach integer multiples of π with increasing number of order.

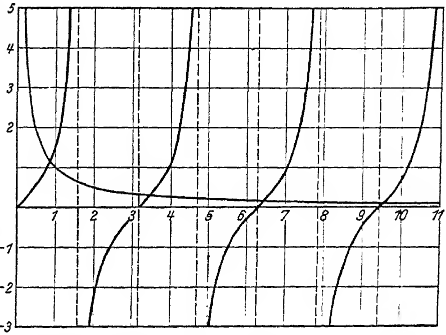

Fig. 98. Transcendental equation to determine the time constants of the eddy-current rotor.

The corresponding natural frequencies p_ν become

$$p_\nu = i\omega - \frac{1}{T_\nu}, \tag{50}$$

where the time constants T_ν can be calculated by equation

$$T_\nu = T \cdot \frac{\coth \frac{\pi\delta}{\tau}}{\frac{\tau}{\pi\Delta}\cdot \nu^2 + \frac{\pi\Delta}{\tau}} \tag{50a}$$

from the time constant $T = 4\tau \cdot 10^{-9} k\Delta \tanh \frac{\pi}{\tau}\delta$ of the normal machine without eddy-current rotor. We may easily demonstrate that the time constants T_ν of higher order become smaller than T and, therefore, conclude that the alternating current impulse of the machine with eddy-current rotor decays more rapidly than the current impulse of the normal machine.

The magnitude of the alternating current impulse follows from equation (44) with consideration of the characteristic values, equation (49a):

$$A_{a.c.} = \mathrm{E}_0 \cdot \frac{1}{4\tau \cdot 10^{-9}} \tanh \frac{\pi\delta}{\tau} \cdot e^{i\omega t} \sum_\nu \frac{e^{T_\nu}}{i\omega - \frac{1}{T_\nu}} \cdot (-T_\nu)$$

$$\frac{\cos\nu - \frac{\tau}{\pi\Delta}\cdot \nu \cdot \coth \frac{\pi\delta}{\tau}\cdot \sin\nu}{\frac{d}{dp}\left(\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\cosh\frac{\pi\Delta}{\tau} + \sqrt{1+\frac{\tau^2}{\pi^2}k^2}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\sinh\frac{\pi\Delta}{\tau}\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\right)} \tag{51}$$

Here,

$$\frac{d}{dp}\left(\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\cdot\cosh\frac{\pi\Delta}{\tau}+\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\sinh\frac{\pi\Delta}{\tau}\sqrt{1+\frac{\tau^2}{\pi^2}k^2}\right.$$

$$\equiv\frac{d}{dp}\left(\cos\nu-\frac{\tau}{\pi\Delta}\tanh\frac{\pi\delta}{\tau}\cdot\nu\cdot\sin\nu\right)$$

$$=\left(-\sin\nu-\frac{\tau}{\pi\Delta}\cdot\tanh\frac{\pi\delta}{\tau}\cdot\sin\nu-\frac{\tau}{\pi\Delta}\cdot\nu\cdot\tanh\frac{\pi\delta}{\tau}\cdot\cos\nu\right)\cdot\frac{d\nu}{dp}$$

$$=\left\{\left(1+\frac{\tau}{\pi\Delta}\cdot\tanh\frac{\pi\delta}{\tau}\right)\cdot\sin\nu+\frac{\tau}{\pi\Delta}\cdot\nu\cdot\tanh\frac{\pi\delta}{\tau}\cdot\cos\nu\qquad\frac{\coth\frac{\pi\delta}{\tau}}{\frac{\tau}{\pi\Delta}\cdot 2\nu}\cdot T\right.$$

$$=\left[\frac{\sin\nu}{2\nu}\cdot\left\{\frac{\pi\Delta}{\tau}\cdot\coth\frac{\pi\delta}{\tau}+1\right\}+\frac{\cos\nu}{2}\right]\cdot T=\frac{T}{2}\cdot\left[\frac{1}{\cos\nu}+\frac{\sin\nu}{\nu}\right].$$

Therefore, respecting equation (49a) we obtain the final result:

$$A_{a.c.}=\mathrm{E}_0\cdot\frac{1}{4\tau\cdot 10^{-9}}\tanh\frac{\pi\delta}{\tau}\cdot e^{i\omega t}\cdot\sum_\nu\frac{e^{-\frac{t}{T_\nu}}}{i\omega-\frac{1}{T_\nu}}\cdot\frac{T_\nu}{T}\cdot\frac{1}{\sinh^2\frac{\pi\delta}{\tau}}\cdot\frac{2\cos\nu}{\frac{1}{\cos\nu}+\frac{\sin\nu}{\nu}}.\tag{51a}$$

For the machine without eddy-current rotor we would obtain ($\Delta\longrightarrow 0$, $k\longrightarrow\infty$, $\Delta\cdot k=$finite):

$$A_{a.c._0}=\mathrm{E}_0\cdot\frac{1}{4\tau\cdot 10^{-9}}\cdot\tanh\frac{\pi\delta}{\tau}\cdot e^{i\omega t}\cdot\frac{{}^{\frac{t}{T}}}{\left(i\omega-\frac{1}{T}\right)\sinh^2\frac{\pi\delta}{\tau}}\tag{52}$$

The amplitudes of the components [equation (51a)] as ratios to the above amplitude are:

$$i_{a.c._\nu}\,{}_{\max}=\frac{T_\nu}{T}\cdot\frac{2\cos\nu}{\frac{1}{\cos\nu}+\frac{\sin\nu}{\nu}}=\frac{T_\nu}{T}\cdot\frac{1+\cos 2\nu}{1+\frac{\sin 2\nu}{2\nu}}$$

$$\frac{\coth\frac{\pi\delta}{\tau}}{\frac{\tau}{\pi\Delta}\nu^2+\frac{\pi\Delta}{\tau}}\cdot\frac{1+\cos 2\nu}{1+\frac{\sin 2\nu}{2\nu}}\tag{53}$$

For example, for $\Delta = \delta$ and $\frac{\pi o}{\tau} = 10$ per cent we find from Fig. 98 for ν as given in equation (49a) and for the ratio of amplitudes $\left|\frac{A_{a.c.}}{A_{a.c.0}}\right|_{\max}$ the following values:

ν	.85	3.42	6.42	9.55	$\frac{8\pi}{2} = 12.55$
$\cos 2\nu$	$-.1288$	.8473	.9582	.976	1
$\sin 2\nu$	.9917	.5312	.2860	.218	≈ 0
$\dfrac{1+\cos 2\nu}{1+\dfrac{\sin 2\nu}{2\nu}}$	.55	1.71	1.91	1.95	≈ 2
$\dfrac{T_\nu}{T}$	1.37	.0855	.0243	.0110	.00635
$\left\lvert\dfrac{A_{a.c.\nu}}{A_{a.c.0}}\right\rvert_{\max}$	75	14.6	4.6	2.2	1.3

For higher numbers of order the approximation of equation (53) does not hold any longer, as we cannot cancel $\frac{1}{T_\nu}$ against $i\omega$; we readily perceive, however, that the stated terms of the series already furnish a practically sufficient picture of the process (Fig. 99).

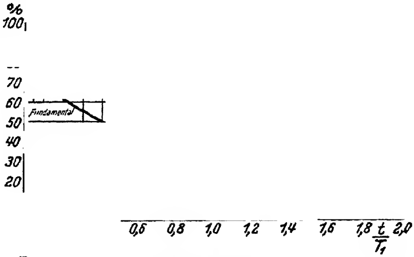

Fig. 99. Course of the alternating current component of the impulse current of a machine with eddy-current rotor.

Of particular interest is the here discussed transient process in connection with the limiting case $\Delta \to \infty$, which comes close to the behavior of eddy-current brakes, or of machines with solid rotors. The summation over the isolated singularities then goes over in an integration along a branch-cut, extending from $p = i\omega$ parallel to the negative real axis up to infinity. The integration may be carried out according to the methods as described above and furnishes simple estimation formulas for the course of the impulse current, these formulas giving the same qualitative results as the above calculation (Fig. 100).*

* See F. Ollendorff: *Arch. Elektrot.*, vol. 24, p. 715 ff.

9. Summary.

The treatment of transient phenomena by means of procedures of function theory is based on the representation of impulse processes by complex integrals; a few fundamental impulse types, in particular, may be described in a simple manner by this theory: the constant impulse, the temporarily constant impulse, the exponential impulse and the oscillating impulse. This method is used to calculate technically important heating and eddy-current phenomena, satisfying the differential equation of heat conduction.

First, the equation of heating of an electric machine is calculated for the case that the heat is conducted away only by the iron core. The problem leads to an integral of a many-valued function, which integral may be transferred to the branch-cut of a double-sheeted Riemann surface. A closed expression for the space-time course of the temperature field is obtained, which expression, in particular, indicates that the winding temperature increases unlimitedly.

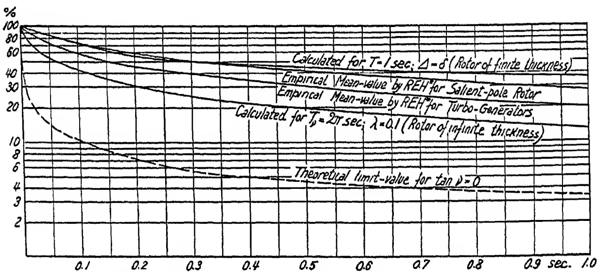

Fig. 100. Calculated and observed course of the alternating current component of the impulse current.

The consideration of external cooling leads to a new equation of heating, which now comprises only the integral of a one-valued function. This integral may be evaluated by means of the residue method. The resulting heating laws all indicate a non-exponential temperature course; for large time values, however, the heating approaches asymptotically an exponential law.

When calculating the heating curve of a machine with internal cooling, an integral on the Riemann surface has again to be computed. Transferring this integral to the branch-cut of this surface, an asymptotic development of the temperature function is derived. A closed summation

* Translator's remark: REH is an abbreviation used by the Society of German Electrical Engineers (V.D.E.) and stands for: "Regulations for the construction, testing and application of high voltage alternating current apparatus."

of this development may be performed for the winding temperature; the resulting heating law cannot be represented by an exponential curve.

Formally, the same computation methods have to be used for the investigation of eddy-current transient processes. As an example, the impulse current of a rotating field eddy-current rotor machine is chosen, the current being composed of an infinite series of free rotating field components. The obtained switching law can be generalized on machines with solid rotor of arbitrary thickness; it satisfactorily checks practical observations.

We refer the reader to the following textbook: C. R. Carson, "Elektrische Ausgleichsvorgänge und Operatorenrechnung" (Berlin: Julius Springer, 1929).

E. SPREADING OF ELECTRIC WAVES ALONG THE EARTH

BY F. NOETHER, BRESLAU

1. The Problem.

The mathematical treatment of the following problem is a typical example of the application of the methods of function theory to physical phenomena pertaining to the spreading of waves. We confine ourselves to a definite problem: the spreading of electro-magnetic waves, issuing from a vertical antenna, along the earth. To simplify matters, let us take no account of the spherical shape of the earth, thus replacing it by a homogeneous body of uniform conductivity and dielectric constant.

As for the rather extensive literature on problems of this nature, we refer to the presentation by A. Sommerfeld in Riemann-Weber (Frank-Mises): "Die Differential-und Integralgleichungen der Mechanik und Physik," vol. 2, p. 542ff. (1927), the author being the originator of the first mathematical treatment of the problem;* this problem has been later methodically changed by H. Weyl.† An essential discrepancy in the result was found,‡ which Weyl calls an "unsatisfactory adjustment of the Sommerfeld-method to the problem." In connection with Sommerfeld's discussion, further papers§ appeared taking into account the spherical shape of the earth as well as the directional effects (horizontal

* "Über die Ausbreitung der Wellen in der drahtlosen Telegraphie," *Ann. Physik*, Ser. 4, vol. 28 (1909), p. 665.

† "Ausbreitung elektromagnetischer Wellen," *Ann. Physik*, Ser. 4, vol. 60 (1919), p. 481.

‡ Compare Weyl's paper, p. 488.

§ Compare Riemann-Weber's book.

antenna). In connection with Weyl's paper, M. I. O. Strutt* has discussed the influence of the altitude of the antenna above ground. In the following representation we shall not adapt the methods of any of the above authors but rather follow a mean line, which it seems is most suited to point out the discrepancies, and starting from which it shall not be difficult to take position to the various methods.

The discrepancies between the different papers are caused by the following question: does the spreading of the energy occur in form of "surface waves," which are restricted to a thin layer near the surface of the earth and which, therefore, because of the cylindrical spreading, may possess a longer transmitting distance, or does the spreading occur in space? By virtue of his investigations, Sommerfeld has concluded the spreading to be essentially of the nature of surface waves; we believe, however, this to be an error in the discussion, although we consider his method, in contrast to Weyl, to be more adapted for special problems. In contrast to Sommerfeld's results, we actually observe long transmitting distances in short wave tests, which are explainable by space waves and their reflection at the "*Heaviside-layer.*"†

2. Basic Statements and Boundary Conditions.

The starting point for the mathematical treatment is furnished by the fact that the electro-magnetic field is determined in air space as well as in the earth by the *wave equation* which follows from Maxwell's equations. The above equation is satisfied by the field components themselves as well as by their potential Π, the so-called *Hertz-function:*

$$\epsilon\frac{\partial^2\Pi}{\partial t^2}+\sigma\frac{\partial\Pi}{\partial t}=c^2\Delta\Pi; \qquad \Delta\Pi=\frac{\partial^2\Pi}{\partial x^2}+\frac{\partial^2\Pi}{\partial y^2}+\frac{\partial^2\Pi}{\partial z^2}, \tag{1}$$

where for air space the dielectric constant is $\epsilon_1=1$ and the conductivity $\sigma_1=0$, while inside the earth ϵ_2 and σ_2 denote the corresponding constants of the earth. If we, furthermore, consider only harmonically-periodic processes of the frequency ω, processes which thus are characterized in all field quantities by the factor $e^{-i\omega t}$, equation (1) goes over into the well-known equation:

$$\Delta\Pi+k^2\Pi=0, \tag{2}$$

*"Strahlung von Antennen," *Ann. Physik,* Ser. 5, vol. 1 (1929), pp. 721, 751; Ser. 5, vol. 4 (1930), p. 1.

†In a continuation of the referred theoretical papers (*Ann. Physik*, Ser. 5, vol. 9 (1931), p. 67). M. I. O. Strutt has measured directly the wave field of a short wave transmitter up to distances of more than 10 wave-lengths with variable ground conditions, finding a confirmation of the theoretical results. These tests again confirm the space wave conception, as Strutt's form of the theory does not contain Sommerfeld's cylindrical waves.

where

$$k^2 = k_1^2 = \frac{\omega^2}{c^2} \text{ in air; } k^2 = k_2^2 = \frac{\epsilon_2\omega^2 + i\sigma_2\omega}{c^2} = k_1^2\left(\epsilon_2 + i\,\frac{\sigma_2}{\omega}\right) \text{ in the earth.}$$

The primary antenna field, i.e. the field without taking into account the earth currents, has been consistently represented in the above mentioned papers as that of a *Hertz-dipole* (Fig. 101). Such a field is understood to be the field created by two electrically charged spheres which are moved closely together in the antenna direction (the vertical direction) and which are connected by a short section of the current, so that an electric oscillation may set up in the system. If the wave length of the oscillation is large as compared with the length of the antenna, the antenna may be represented by a single dipole. In the case of very short waves, or complicated antenna forms, the finite length of the antenna should be represented by a succession of dipoles.* As did Sommerfeld and Weyl, we also confine ourselves on the first case. Let $p_0 = \rho \cdot l$ be the electric momentum of the dipole, periodic in time (ρ = charge, l = length), the wave length in air of the electric oscillation be $\frac{2\pi}{k_1} = \frac{2\pi c}{\omega}$ and $R = \sqrt{x^2+y^2+z^2}$ be the distance between an arbitrary variable point A and the dipole center. Then, the electric and magnetic primary field of the antenna is represented by the following spheric-symmetrical solution of the wave equation:

$$\Pi_0 = p_0 \cdot \frac{e^{ikR}}{R} \tag{3}$$

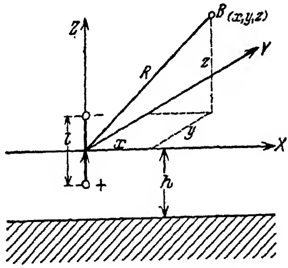

Fig. 101. Orientation at the dipole.

$\frac{\Pi_0}{\epsilon l}$ would represent the wave-form potential of a single dipole, from which $\frac{l}{\epsilon}\frac{\partial \Pi_0}{\partial z}$ follows as the electric potential of the dipole. Furthermore, $\frac{\partial \rho_0}{\partial t} = -i\omega\rho_0$ is the antenna current in the conductor section of length l, the current creating the magnetic field; accordingly, by Biot-Savart's law, $-i\omega\Pi_0$ is the "vector potential" the rotation of which determines the magnetic field.† It is characteristic and of significance for the following boundary conditions, that in the electric potential the variable denominator ϵ appears (i.e. variable with

* Compare Strutt's papers.

† For details see, for example, Abraham-Föppl: "Theory of Electricity," vol. 1, sect. 3.

the medium), while in the magnetic potential no such denominator appears.

Let us at first assume (in agreement with Weyl) that such a primary dipole is located in the air at the height h above earth, so that the separating plane earth-air may be considered as the plane $z=-h$ in Fig. 101. The preliminary assumption made by Sommerfeld, that the dipole is situated in the separating plane itself, would lead to certain complications which have not been sufficiently accounted for in his representation. We shall be able to express the modifications of the field due to the addition of the earth by means of another secondary Hertz function Π_1, in air and Π_2 in the earth, which functions again have to satisfy the wave equations (2) and, at that, must not contain any singularity. The magnetic and electric additional field produced by the separating plane (reflection and refraction) is expressed by Π_1 and Π_2 in the same manner as the above primary field was expressed by Π_0; along the separating plane $z=-h$, therefore, the following boundary conditions have to be satisfied:

1. Continuity of the tangential electric field components or, in other words, of the electric potential, i.e.:

$$\frac{1}{k_1^2}\left(\frac{\partial \Pi_0}{\partial z}+\frac{\partial \Pi_1}{\partial z}\right)=\frac{1}{k_2^2}\frac{\partial \Pi_2}{\partial z} \qquad \text{for } z=-h. \tag{4a}$$

Here, the denominator k^2 is introduced in place of the above mentioned denominator ϵ, k^2 being proportional to ϵ.

2. Continuity of the tangential magnetic field components or, in other words, of the magnetic vector potential, i.e.:

$$\Pi_0+\Pi_1=\Pi_2 \qquad \text{for } z=-h. \tag{4b}$$

3. Analytic Representation of the Primary Antenna Field.

The next step is the creation of a representation (used by Weyl) of the function Π_0, directly permitting the fulfillment of the above boundary conditions. In order to secure this representation, let us start from the *plane* wave distributions, satisfying equation (2) (with $k=k_1$). Such a plane wave is expressed by:

$$e^{ik(\alpha x+\beta y+\gamma z)}$$

where α, β, γ are the direction cosine of the wave normals, for which the relation

$$\alpha^2+\beta^2+\gamma^2=1$$

holds. We first perform the summation of such waves which have equal amplitudes for all directions of space; this means that we consider the

directional factors α, β, γ to be points of a unit sphere and introduce on the latter spheric angles ϑ and φ, the orientation of which is towards the vertical z-axis. Let a surface element on this sphere be denoted by df. Thus:

$$\alpha=\sin\vartheta\cos\varphi, \qquad \beta=\sin\vartheta\sin\varphi, \qquad \gamma=\cos\vartheta, \qquad df=\sin\vartheta\, d\vartheta\, d\varphi.$$

Hence, the result of the summation is:

$$S=\frac{1}{2\pi}\int_{\text{Sphere}} e^{ik(\alpha x+\beta y+\gamma z)}\,df$$

$$=\frac{1}{2\pi}\int_{-\pi}^{+\pi} d\varphi \int_0^{\pi} e^{ik[(x\sin\varphi+y\cos\varphi)\sin\vartheta+z\cos\vartheta]}\sin\vartheta\, d\vartheta. \tag{5}$$

This integral, surely showing spherical symmetry also with reference to the coördinates x, y, z and, thus, depending only on R, it is sufficient to perform the evaluation for the one case $x=y=0$, $z=+R$, and carrying out the integration with respect to φ first, we obtain at once:

$$S=\int_0^{\pi} e^{ikR\cos\vartheta}\sin\vartheta\, d\vartheta=\int_{-1}^{+1} e^{ikRu}\,du=\frac{e^{ikR}-e^{-ikR}}{ikR}=2\,\frac{\sin kR}{kR}. \tag{5a}$$

The field calculated in this manner thus does not represent a spreading of waves away from the radiation source $R=0$, because the field function should there become infinite. We rather obtain — as was to be expected from Art. 2, where we have been dealing with waves in all directions — a state of stationary wave distribution composed of incoming and outgoing wave radiation such that the function S remains finite at $R=0$. But the required radiating part $\dfrac{e^{ikR}}{ikR}$ is contained in formula (5a) as its first term and may be separated by decomposing the path of integration with respect to u into 2 paths: first path from $u=-1$ to $u=+i\cdot\infty$ (so that the expression under the integral sign vanishes there), second path from $u=+i\cdot\infty$ to $u=+1$. The last part gives:

$$\frac{e^{ikR}}{ikR}=\int_{+i\cdot\infty}^{+1} e^{ikRu}\,du=\int_0^{\frac{1}{2}\pi-i\cdot\infty} e^{ikR\cos\vartheta}\sin\vartheta\, d\vartheta, \tag{5b}$$

while the first part gives $-\dfrac{e^{-ikR}}{ikR}$ in equation (5a).

This seems to suggest the performance of a corresponding decomposition of the more general integral (5) itself. The corresponding part integral is thus first brought into the form:

$$S_1 = \frac{1}{2\pi} \int_{-\pi} d\varphi \int_{0} e^{ik[(x \sin \varphi + y \cos \varphi) \sin \vartheta + z \cos \vartheta]} \sin \vartheta \, d\vartheta, \tag{6}$$

where the upper limit A, to be determined later, is to be chosen such that the exponent of the function under the integral sign possesses there a negative infinite real component, the function under the integral itself thus approaching 0. This is actually possible, and, moreover, the spherical symmetry of the solution representing this integral is also maintained.

To explain this clearly, let us write in polar coördinates:

$$x = r \cos \chi, \qquad y = r \sin \chi, \qquad (r = \sqrt{x^2 + y^2}).$$

Referring to the exponent in equation (6), we then have: $x \cos \varphi + y \sin \varphi = r \cos (\varphi - \chi)$, so that, keeping the limits, we may integrate over the angle $\psi = \varphi - \chi$ instead of over φ. This eliminates a dependency of χ. The limits for ψ are given by a complete cycle of $\cos \psi$; but we also may transfer this path into the complex ψ-plane. To make this possible, we first assume $r \sin \vartheta$ to be positive real and also z to be positive. We then may fix the (still open) integration path $0 - A$ in the ϑ-plane as indicated in Fig. 102. Let now $\psi = v + iw$ and thus,

$$\cos \psi = \cos v \cosh w - i \sin v \sinh w.$$

We see from this equation, that under the assumptions just made, the integration path in the ψ-plane may, by Cauchy's integral theorem, be transposed in accordance with Fig. 103, i.e. may be composed of the two sections I and II. The cross-hatchings indicate the stripes in which these paths are allowed to reach infinity, i.e. where the real part of $i \cos \psi$ is negative infinite.*

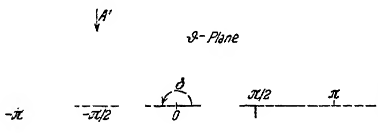

Fig. 102. Integration path in the ϑ-plane.

Now, also path II may

* This decomposition, to be sure, is not possible in the case $r=0$, because then both integrals I and II, taken individually, would become infinite and only their sum would remain finite.

be replaced by path I if, at the same time, the sign of ϑ is reversed. If we, namely, move in the ϑ-plane from the positive to the negative real axis along a small semicircle, i.e. counterclockwise about the origin (Fig. 102), thus putting $r \sin \vartheta = \rho e^{i\delta}$ (δ from 0 to π), the imaginary part of $\cos \psi \cdot r \cdot \sin \vartheta$ becomes:

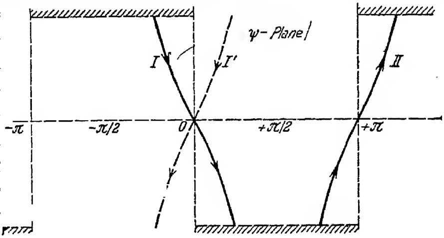

Fig. 103. Integration path in the ψ-plane.

$$\rho = \text{Im}\,[(\cos \delta + i \sin \delta)(\cos v \cosh w - i \sin v \sinh w)]$$
$$= \rho(-\sin v \cos \delta \sinh w + \cos v \sin \delta \cosh w),$$

or, with sufficiently large positive w:

$$-\tfrac{1}{2}\,\rho\, e^{w} \sin (v - \delta)$$

and with sufficiently large negative w:

$$\tfrac{1}{2}\rho\, e^{w} \sin (v + \delta).$$

This states, that with increasing δ the *upper* cross-hatched regions of Fig. 103 shift to the *right*, the *lower* ones to the *left*, the total shifting amounting to π. The integration path I goes over in the path I' (Fig. 103), the latter coinciding with path II, if $r \sin \vartheta$ is again chosen positive, $\cos \psi$, however, negative (i.e. ψ is increased by π). It will be noted that the direction of integration reverses because of the factor $\sin \vartheta$. But this transformation just means the following: we may restrict the integration in the ψ-plane (Fig. 103) to the path I (with I' for negative ϑ), if we supplement in the ϑ-plane (Fig. 102) the path A, by means of the additional path A', to an infinite path. The carrying out of the ψ-integration along the path I or I' would result in so-called *Hankel-functions* of first kind which we, however, do not introduce for the present. Integration along II, on the contrary, would lead to Hankel-functions of second kind which may be replaced by such of first kind with negative arguments.* Thus, we have to treat the line integral along I and the line integral along $A' - A$, i.e. the expression:

$$S_1 = \frac{1}{2\pi}\int d\psi \int^{A} e^{ik(r \cos \psi \sin \vartheta + z \cos \vartheta)} \sin \vartheta \, d\vartheta. \tag{7}$$

The conversion was necessary for the further discussion of the boundary-

* Compare A. Sommerfeld in Riemann-Weber's book, vol. 2, pp. 454, 549.

value problem; this conversion first shows that the integral S_1 is also spherical symmetric as is the integral S, i.e. it is a function of R alone. This results from the following: the infinite integration paths in the ψ-plane and in the ϑ-plane go again over in infinite paths, if the spheric coördinates ψ, ϑ are transformed into new ones ψ', ϑ' by means of formulas of spheric trigonometry (see Fig. 104):

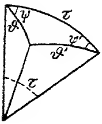

Fig. 104. Transformation of space angles.

$$\cos\vartheta' = \cos\vartheta\cos\tau + \sin\vartheta\sin\tau\cos\psi;\quad \sin\psi' = \frac{\sin\psi\sin\vartheta'}{\sin\vartheta}$$

The transfer may take place continuously, the regions of validity overlapping in such a way, that a finite region is always available for the transfer of the old integration paths to the new ones. If the transformation angle τ denotes the angle between R and z so that the axis $\vartheta'=0$ coincides with the direction R itself, i.e. if

$$z = R\cos\tau, \qquad r = R\sin\tau, \tag{8}$$

the exponent in equation (7) becomes independent of ψ', viz. $ikR\cos\vartheta'$. The new integration over the angle ϑ' is thus reduced to the integration already carried out above, yielding as there, but now for *any* direction τ, the value

$$S_1 = \frac{e^{ikR}}{ikR}$$

This method of proof was indicated by Weyl and seems to be best adapted to point out the fact of the spherical symmetry of the solution. As, however, in the further treatment of the problem (boundary conditions), the planes $z=\text{const}$ are distinguished by physical reasons, the choice of coördinates as suggested by Weyl and as demonstrated by Strutt would lead to cumbersome formations of mean values. The arrangement as suggested by Sommerfeld, which corresponds to the retention of the coördinates ψ and ϑ, appears to be more appropriate, and we, therefore, accept this arrangement, modifying it correspondingly.

4. Fulfillment of Boundary Conditions.

The performed decomposition of the basic function $\Pi_0 = \frac{1}{R}e^{ikR}$ into linear solutions which may be interpreted as plane waves, is particularly suited to fulfill the boundary conditions (4a) and (4b). For, to do this, it is only required to add a reflected and a refracted plane wave to the primary-incident one, in analogy with the well-known derivation of Fresnel's equations in optics. The distinction of "incident" and "reflected" wave has here, where we deal with generally complex

ϑ-values, only the meaning of an illustrative expression for two solutions of the wave equations, differing in the sign of $\cos\vartheta$ and having $\sin\vartheta$ in common. To fulfill the boundary conditions it appears advisable to go back once more to the limits of integration of equation (6) $[-\pi<\psi<+\pi;\ 0<\vartheta<A]$ and to use, of course, the expression under the integral sign in the form of equation (7). But as the origin of the system of coördinates lies in the antenna and, thus, the surface of the earth is given by $z=-h$, we need a representation of the primary field for negative values of z. This we obviously obtain by a change of sign in the exponent as follows:

$$p_0\frac{e^{ik_1R}}{ik_1R}=\frac{\Pi_0}{ik}=\frac{p_0}{2\pi}\int_{-\pi}^{+\pi}d\psi\int_0^A e^{ik_1(r\cos\psi\sin\vartheta-z\cos\vartheta)}\sin\vartheta\,d\vartheta. \tag{9}$$

These limits have the advantage that they may be kept fixed for all r-values in the plane $z=-h$, i.e. by equation (8), as long as $\frac{1}{2}\pi<\tau\leqq\pi$; this is necessary to be able to fulfill the boundary conditions of a plane wave along the complete plane, including the point $r=0$ (i.e. $\tau=\pi$). The path A in the ϑ-plane can remain unchanged for all values of the integration variable ψ from $-\pi$ to $+\pi$, i.e. for positive and negative real $r\cos\psi$-values, so that the sequence of the integrations of equation (9) is immaterial.

Under the integral sign we now have:

Incident wave: $\dfrac{p_0}{2\pi}e^{ik_1(r\cos\psi\sin\vartheta-z\cos\vartheta)}$. (10)

Reflected wave: it differs from the incident wave in the sign of $\cos\vartheta$ and in its amplitude and thus may be written:

$$\frac{p_1}{2\pi}e^{ik_1[r\cos\psi\sin\vartheta+(z+2h)\cos\vartheta]}. \tag{10a}$$

This part, as shown by its form, permits the above integration path for $z>-2h$.

The "*refracted*" wave, finally, differs from the incident one by the fact that k_2 replaces k_1 and a refraction angle η replaces the incidence angle ϑ and aside of that — by the amplitude factor. This wave may, hence, be written:

Refracted wave: $\dfrac{p_2}{2\pi}e^{ik_2[r\cos\psi\sin\eta-(z+h)\cos\eta]+ik_1h\cos\vartheta}$ (10b)

The exponents in equations (10) to (10b) are written in such a manner, that they are identical along the plane $z=-h$ if

$$k_1\sin\vartheta=k_2\sin\eta, \tag{11}$$

i.e. if the refraction law holds; the boundary conditions (4a) and (4b) then require the relations:

$$\frac{\cos\vartheta}{k_1}(p_0-p_1)=\frac{\cos\eta}{k_2}p_2, \tag{11a}$$

$$p_0+p_1=p_2, \tag{11b}$$

from which follows:

$$p_1=p_0\frac{k_2\cos\vartheta-k_1\cos\eta}{k_2\cos\vartheta+k_1\cos\eta}; \qquad p_2=p_0\frac{2k_2\cos\vartheta}{k_2\cos\vartheta+k_1\cos\eta}. \tag{12}$$

After having determined those quantities, we now may move the primary dipole into the separating plane itself, i.e. consider the limiting case $h=0$. This is, by the way, not essential, but is done in the following because of increased simplicity. The discussion of the conditions in air, $z\geqq0$, is, furthermore, typical and sufficient. In the air space the reflected wave is already given by equations (10a) and (12), and the path of integration may remain unchanged. For the primary wave we only have, by equation (6), to replace $-z\cos\vartheta$ by $+z\cos\vartheta$ in the exponent, which thus becomes identical to the one of the reflected wave for $h=0$. By equation (10a) the following relation, therefore, holds in the region $z\geqq0$:

$$\Pi_0+\Pi_1=\frac{ik_1}{2\pi}\int_{-\pi}^{+\pi}d\psi\int_0^A(p_0+p_1)e^{ik_1(r\cos\psi\sin\vartheta+z\cos\vartheta)}\sin\vartheta\,d\vartheta.$$

The transformation of the path of integration may here be accomplished in accordance with equation (7), so that the paths of integration become infinite in ϑ and ψ. Substituting equations (11b) and (12) into the above expression we obtain (compare also Figs. 102 and 103):

$$\Pi_0+\Pi_1=\frac{ik_1k_2p_0}{\pi}\int_I d\psi\int_{A'}^{A}\frac{\cos\vartheta\sin\vartheta}{k_2\cos\vartheta+k_1\cos\eta}e^{ik_1(r\cos\psi\sin\vartheta+z\cos\vartheta)}d\vartheta. \tag{13}$$

The formula found in this manner goes essentially over in the one upon which Sommerfeld based his discussion,* if we first carry out the integration with respect to ψ and then choose $\lambda=k_1\sin\vartheta$ as the second integration variable. As a matter of fact, the derivation used by Sommerfeld is a different one, as he assumed $h=0$ to start with. We have, however, to consider that the form of equation (13) only holds for $z\geqq0$, this equation thus not fulfilling directly the boundary condition (4b). To satisfy this condition in a one-valued sense the continuation of the solution into the region $z\leqq0$ was necessary, which corresponds

* Compare: A. Sommerfeld in Riemann-Weber's book, p. 550.

to our method of derivation. Sommerfeld's reasoning, avoiding this fact by a direct change of the boundary condition, therefore appears to us to be less urgent.

5. Discussion of the Solution.

As for the rest of the discussion, we shall have to replace suitably the integration path $A'-A$. Sommerfeld has pointed out the significance of the denominator under the integral sign,

$$D=k_2 \cos \vartheta+k_1 \cos \eta, \tag{14}$$

the zeros of which are of particular interest (Weyl omitted this investigation except for one limiting case). These points, furnishing a pole of the function under the integral sign, indicate, that for the respective incidence angle ϑ_0, even for a vanishing amplitude p_0, there still could appear an amplitude p_1 in the first medium and an amplitude p_2 in the second one, i.e. a "free" spreading of waves could exist even without the impulse of the incident wave. This case finds its analogy in the polarization angle of optics: there too only *one* wave, the incident one, exists in the first medium and another wave, the refracted one, in the second medium. In our case, ϑ_0 denoting a complex angle, the waves are of a different type: they are "non-homogeneous" waves or "surface waves" and represent the simplest form of such waves in the realm of electricity. Better known, but more complicated, are such waves in the form of transmission line waves, propagating in air along a transmission line; the amplitude of such waves decreases rapidly from the conductor surface towards the inside of the conductor and slowly so towards the outside. This non-homogeneity has its reason in the energy dissipation by the ohmic resistance of the conductor. In our case, the conducting surface of the earth replaces the conductor surface.

By equation (14) in conjunction with the law of refraction (11), the equations for the determination of the angle ϑ_0 and of the corresponding refraction angle η_0 are:

$$k_2 \cos \vartheta_0+k_1 \cos \eta_0=0; \qquad k_1 \sin \vartheta_0-k_2 \sin \eta_0=0, \tag{15}$$

from which follows:

$$\sin 2\vartheta_0+\sin 2\eta_0=0,$$

$$\vartheta_0+\eta_0=m\pi \quad \text{or} \quad \vartheta_0-\eta_0=(2m+1)\frac{\pi}{2},$$

where m is an integer. The first cases are eliminated because they would have to require $\frac{k_2}{k_1}=\pm 1$ by equation (15). The other cases, by equation

(15), result in the so-called *Brewster's law:*

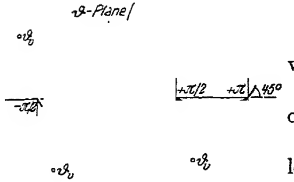

Fig. 105a. Branch points in the ϑ-plane.

$$\tan \vartheta_0 = \pm \frac{k_2}{k_1}, \tag{16}$$

which, in this general form, permits one root in each stripe of width $\frac{1}{2}\pi$ located in the complex ϑ-plane (we call it a "quadrant"; compare Fig. 105a). For the following discussion of our integration path, however, one-half of these roots is eliminated to begin with. For, the law of refraction (11) fixes $\sin \eta$ uniquely for given $\sin \vartheta$, but $\cos \eta$ only with exception of the sign, i.e. the points ϑ_v* of the ϑ-plane, where

$$\cos \eta_v = \pm \sqrt{1 - \frac{k_1^2}{k_2^2} \sin^2 \vartheta_v} \tag{17}$$

vanishes, are to be considered as "branch points" of the integral (13), $\cos \eta$ changing its sign when we complete a circuit through these points. In equation (13) the sign of $\cos \eta$ is assumed such that, for instance, $\eta = 0$ (not $\pm\pi$) for $\vartheta = 0$ and thus $\cos \vartheta = \cos \eta = +1$. We only have to consider the cases which follow from the above without circuiting of the branch points, i.e. the cases for which η lies in the neighborhood of 0 (not at $\pm\pi$), as $\frac{k_2^2}{k_1^2}$ has a large absolute value. For example, for $m = 0$, i.e. $\vartheta_0 - \eta_0 = \frac{1}{2}\pi$, it follows from equation (15): $\tan \vartheta_0 = -\frac{k_2}{k_1}$; ϑ_0 thus could lie in the first negative or in the second positive quadrant. The last case only has to be considered, as in the first case $\eta_0 = \vartheta_0 - \frac{1}{2}\pi$ would lie in the second negative quadrant, i.e. near $-\pi$. Analogously it follows for $m = -1$, i.e. $\vartheta_0 - \eta_0 = -\frac{1}{2}\pi$, that ϑ_0 has to lie in the second negative quadrant. For larger m-values these positions repeat themselves with the cycle 2π.

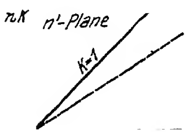

Fig. 105b. Referring to the position of the branch points.

Let, furthermore, (in the notation as used in optics)

$$\frac{k_2}{k_1} = \sqrt{\epsilon_2 + \frac{i\sigma_2}{\omega}} = n(1 + i\kappa) = n',$$

* Translator's note: The subscript "v" stands for the German word "Verzweigungspunkt," meaning "branch point."

where n designates the "refraction index" and κ the "absorption index." The latter is in our case <1, as ϵ_2 and σ_2 are both positive. Equation (16) transacts a simple mapping of the complex n'-plane upon the ϑ-plane, each ray $\kappa = \text{const}$ of the n'-plane going over in a curve running from $\vartheta = \pm\pi$ to $\vartheta = \pm\frac{1}{2}\pi$; the ray $\kappa = 1$, in particular, goes over into a curve intersecting the axis under 45°, as is indicated in Figs. 105a and 105b. The regions shown in Fig. 105a ones, inside of which, according to the previous statements, lie the discussed poles ϑ_0, the closer to $\pm\frac{1}{2}\pi$ the larger n is.

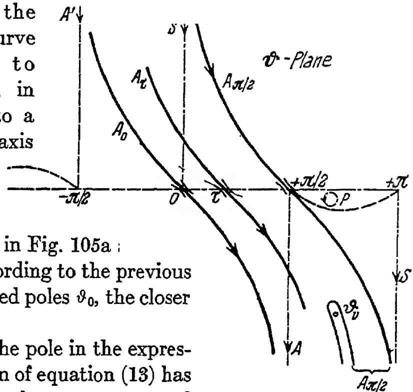

Fig. 106. Saddle-point integration.

After the position of the pole in the expression under the integral sign of equation (13) has been pointed out more exactly, we may proceed to investigate, whether the pole exerts an essential influence on the course of the wave spreading, i.e. whether it exerts a still predominant influence in a sufficiently large distance from the antenna. This then would amount to the appearance of "surface waves" in the sense as defined above. Sommerfeld has expressed this influence by appropriately deforming the path of integration as indicated in Fig. 106 (the path is transferred upon his $\lambda = k_1 \sin \vartheta$-plane); he chooses a deformation into the path S (Fig. 106), which is characterized by the fact that $\cos \vartheta$ is real along the complete path from $+\infty$ to $-\infty$. To carry out this transposition he had, however, to pass a pole (denoted by P in Fig. 106), for which there still remains a residue integral to be calculated by Cauchy's theorem. This residue integral is obtained as follows: after dividing the denominator of the function under the integral sign by $\dfrac{(\vartheta-\vartheta_0)}{2\pi i}$ we simply insert ϑ_0 in place of ϑ, thus directly representing a wave with the incidence angle ϑ_0, i.e. a *surface wave*. Its amplitude results from the indicated calculation and is:

$$p^{(0)} = p_0 \frac{2k_2^2 \cos \vartheta_0 \sin \vartheta_0 \cos \eta_0}{k_2^2 \sin \vartheta_0 \cos \eta_0 + k_1^2 \cos \vartheta_0 \sin \eta_0}. \tag{18}$$

The remaining integral along the path S is interpreted by Sommerfeld

as a *space wave*. (To facilitate the comparison with Sommerfeld's investigation we transfer, in Fig. 107, the paths of Fig. 106 into the plane $\lambda = k_1 \sin \vartheta$ used by him.)

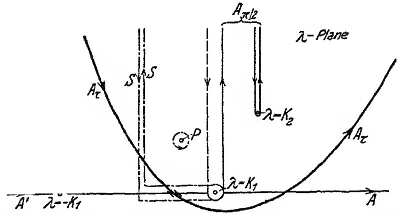

Fig. 107. Referring to the discussion of the wave by Sommerfeld.

6. Asymptotic Calculation of the Solution.

Already Weyl has emphasized the arbitrariness of this decomposition, and it can indeed hardly be realized, why the path S is supposed to characterize the type of a space wave.* Much more obvious appears to be the following method of calculation: The integration path $A' - A$ of Fig. 106 terminates at both ends in regions in which the exponential function under the integral sign moves towards zero, the exponent possessing a negative real component. But not only the factor $\cos \vartheta$ of z should be considered with respect to this exponent (as did Sommerfeld) but also the factor $\sin \vartheta$ of r. Because of

$$r \sin \vartheta + z \cos \vartheta = R \cos (\vartheta - \tau) \tag{19}$$

(compare equation (8)), these bounding regions shift with $\tau = \tan^{-1} \frac{r}{z}$. Between the limits (at both ends) the absolute value of the function to be integrated has of course to rise and then to fall. Riemann introduced a method, the so-called "saddle-point" integration,† in which the path is selected such as to assure rapid rise and fall, possibly in such a manner, that the exponent deviates from its maximum value by negative real values only. This is achieved by first finding the extreme points, i.e. in the case of equation (19) the points $\vartheta - \tau = 0$. In their neighborhood the development is:

$$iR \cos (\vartheta - \tau) = iR[1 - \tfrac{1}{2}(\vartheta - \tau)^2 + \cdots], \tag{19a}$$

and the mentioned sloping curves run such that $-i(\vartheta - \tau)^2$ is negative real, i.e. that $\vartheta - \tau$ takes its course in the complex ϑ-plane under an angle

* In his paper *Ann. Physik* (Ser. 4, vol. 28), Sommerfeld attempts to provide reasons for it by means of a development into series of the function under the integral sign (p. 703). But it is in the neighborhood of the pole P that this development has to become divergent; the series development, therefore, does not in a sufficient degree take account of the influence of the pole upon this part of the integral, which part is directly opposite to the previously mentioned residue.

† See for example: Courant-Hilbert: "Methoden d. math. Physik," vol. 1, p. 436ff.

of 45° (compare Fig. 106). The thus determined saddle-point path A_τ shifts with variable τ parallel to itself and is to be considered between $\tau=0$ (path A_0) and $\tau=\frac{1}{2}\pi$ (path $A_{\pi/2}$). As will be noted, this path touches the pole region from outside (as indicated in Fig. 106) at $\vartheta=\frac{1}{2}\pi$, without, however, penetrating it. Note also the small deformation of the path which is necessary for larger τ-values because of the branch point ϑ_v and which is indicated in Fig. 106 at the right bottom corner. This deformation is without significance for the following method as ϑ_v possesses a large imaginary component (because of $\frac{k_2}{k_1}>1$).

Dealing with the kind of transposition as described above, a residue integral, therefore, does not have to be considered, and it only remains to discuss the result of the saddle-point integral under assumption of sufficiently large R-values. The fact that the result represents the type of pure space waves is realized from the following reasoning: the greater R, the more rapid is the drop of the function under the integral sign from its maximum value and the more, therefore, the values in the neighborhood of the maximum should be taken into consideration when performing the integration. This is the nucleus of the saddle-point method. In first approximation, only the values at the point $\vartheta=\tau$ itself are decisive. In the same manner the primary integral (7) might have been evaluated and would have furnished, as an approximation, for large R-values, its correct value $\frac{e^{ikR}}{ikR}$, this expression being identical with its asymptotic character. But equation (7) furnishes equation (13) when we comprise the factor $\frac{2ik_1k_2p_0\cos\vartheta}{k_2\cos\vartheta+k_1\cos\eta}$ in the function under the integral sign; substituting $\vartheta=\tau$ and expressing $\cos\eta$ by ϑ, the asymptotic result follows in the form:

$$\Pi_0+\Pi_1=\frac{2n^2p_0\cos\tau}{n^2\cos\tau+\sqrt{n^2-\sin^2\tau}}\cdot\frac{e^{ik_1R}}{R}. \tag{20}$$

It represents space waves, the amplitude of which depends on the direction of propagation τ and vanishes for $\tau=\frac{1}{2}\pi$, i.e. in the horizontal direction along the surface of the earth (by interference between the incident and reflected wave). This means, more exactly, that the wave approaches zero in a higher order than according to equation (20), which fact could be disclosed by the next approximation.

It should be added that for the integration with respect to ψ similar considerations hold; $\psi=0$ is then to be considered as a saddle-point. If first integrating with respect to ϑ, r would have to be replaced by $r\cos\psi$,

which does not constitute an essential change as, along the path I (Fig. 103), $\cos\psi$ may be assumed to be positive or at least to have a positive real component. It seems to be most appropriate to apply the asymptotic method simultaneously to both variables, proceeding as follows:

Let us write down equation (13) in the form

$$\Pi_0+\Pi_1$$

$$=p_0\frac{ik_1k_2}{\pi}e^{ik_1R}\int_I d\psi\int_{A'}^{A} e^{-2ik_1R\,[\sin\vartheta\sin\tau\sin^2\frac{1}{2}\psi+\sin^2\frac{1}{2}(\vartheta-\tau)]}\frac{\cos\vartheta\sin\vartheta}{k_2\cos\vartheta+k_1\cos\eta}\,d\vartheta.$$

Let, further,

$$\sin\tfrac{1}{2}\psi\sqrt{2ik_1R\sin\vartheta\sin\tau}=u;\qquad \sin\tfrac{1}{2}(\vartheta-\tau)\sqrt{2ik_1R}=v.$$

Then the integral becomes

$$\Pi_0+\Pi_1$$

$$=p_0\frac{2k_2}{\pi}\frac{e^{ik_1R}}{R}\int\!\!\int_{-\infty}^{+\infty}e^{-(u^2+v^2)}\frac{\cos\vartheta\sqrt{\sin\vartheta}\,du\,dv}{\sqrt{\sin\tau}(k_2\cos\vartheta+k_1\cos\eta)\cos\frac{1}{2}\psi\cos\frac{1}{2}(\vartheta-\tau)}\,. \quad (21)$$

The first approximation then consists in replacing $\cos\frac{1}{2}\psi$ and $\cos\frac{1}{2}(\vartheta-\tau)$ by 1 and ϑ by τ, η being again determined from equation (11), so that we only have to evaluate the integral $\int\!\!\int_{-\infty}^{+\infty} e^{-(u^2+v^2)}\,du\,dv=\pi$. This furnishes formula (20). The continuation would then simply consist in developing the second term of the function under the integral sign with respect to ψ and $(\vartheta-\tau)$, or rather, to u and v, and then in performing the integration.

The final formula (20) indicates that the field distribution at a sufficiently large distance is characterized as a pure space-wave type. The amplitude of this space wave is, by virtue of the variable reflection-ratio, still dependent on the direction. It is true that at a small distance the influence of the pole (further development of equation (21)) will result in a deviation from this pure type; there also will remain a "wave-perturbance" in a purely horizontal direction. But this wave-perturbance can not reach farther than the calculated space-wave distribution, and the spreading of energy, thus, can not occur in the form of surface waves.

From an analytic point of view this fact shows up in the exponential factor of the integral (21), the exponent making the function under the

integral sign on the path A, even in the neighborhood of the pole P, much smaller than on the Sommerfeld-path S. This factor does not only mean an exponential damping produced by earth conductivity, but it is, on the contrary, independent of the latter and is fixed by the constants of the upper air space only. Physical reasonings, too, make it clear that antenna impulses of any form can not spread as surface waves, i.e. cylindrically. For, the transmission-line waves, which served as a model for surface waves, can only be maintained by a difference in voltage between the line and the return line; in the case of the antenna impulse, however, such a potential difference does not exist: it should act between earth and a layer at great height. (We rather would expect the character of cylindrical waves in waves created by lightning, as the conditions here are similar to those of the transmission-line waves, and as we could account for the participation of the Heaviside layer in the process of wave propagation.)

These relations stress another point of interest in the limiting case of very high ground-conductivity σ_2. This case has been discussed in detail by Sommerfeld and also by Weyl. By equation (16), namely, for $k_2 = \infty$ also $\tan \vartheta_0$ becomes infinite, i.e. ϑ_0 moves towards $\frac{1}{2}\pi$ and $\cos \vartheta_0$ as well as k_1/k_2 move towards 0; the amplitude $p^{(0)}$ of the cylindrical wave becomes small by equation (18) and is proportional to $2\, p_0 k_1/k_2$. It is true that this would have to be ascertained in any individual case by means of a slight modification of our asymptotic procedure, the modification being necessary because, for $\tau = \frac{1}{2}\pi$, pole and saddle-point coincide in accordance with Fig. 106. But, as far as the order of magnitude of the surface waves is concerned, the just stated value holds, and, therefore, we believe them to be insignificant even in this limiting case. It should be furthermore noted that the relations directly at the horizontal plane $\tau = 0$ do not provide a satisfactory picture at all; for, theoretically (i.e. as far as an exact fulfillment of the condition of reflection exists directly at the surface of the earth) also the space waves vanish here, by equation (20), due to interference between the incident and the reflected wave, which proves, that at least small elevation angles should be taken into account.

As a final realization, Sommerfeld's most important result remains untouched by the above discussion, the result that high ground conductivity (sea water) is favorable to wave spreading in the horizontal plane $\tau = 0$, which agrees with practical experience. This result has been recently derived in a simpler manner by B. van der Pol.* But this result

* "Über die Ausbreitung elektromagnetischer Wellen," *Jahrb. d. drahtl. Telegraphie u. Telephonie, Zeitschr. f. Hochfrequenztechnik,* vol. 37 (1931), p. 152; extended discussion in *Ann. Physik,* Ser. 5, vol. 10 (1931), p. 485.

is only an expression of the fact that the space-wave maximum comes, by equation (20), with increasing σ_2 closer and closer to the direction $\tau = 0$, finally approaching it discontinuously; this is not in favor of the surface-wave conception of wave spreading, but in favor of the space-wave conception. This, the curves indicated in Fig. 108 and calculated by Strutt* show in a particularly clear manner, the curves at the same time indicating the situation for different heights h of the primary dipole. The curves give the directional distribution of the field intensity of the space-wave spreading as represented by the length of the radius vector in the corresponding direction. Other antenna forms, for example the horizontal antenna, may be investigated by the above method without new difficulties; this was also done by Strutt in a form corresponding to Weyl's choice of coördinates. Such an investigation is, of course, important also in connection with the *transmitting distance* of the electromagnetic waves, it being dependent on the *direction of propagation* because of the reflections at the Heaviside layer.†

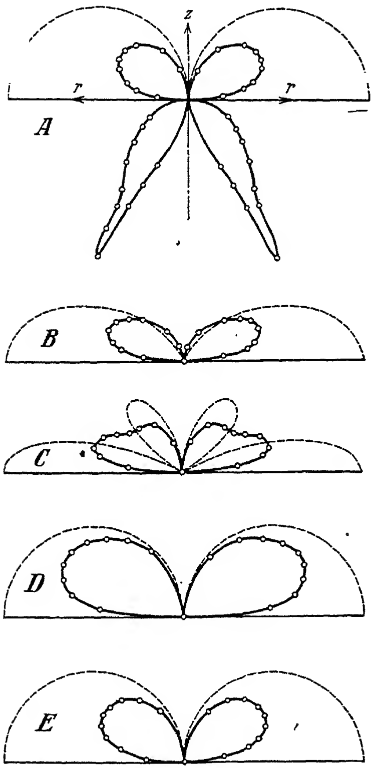

Fig. 108. Wave characteristics (Strutt).
A: Dipole on non-conductive earth ($n'^2=4$).
B: Poorly conducting earth; height of dipole $h=\frac{1}{4}$ wave length.
C: Same; $h=\frac{1}{2}$ wave length.
D: For $n'^2=80$ (water); $h=0$.
E: earth of moderate conductivity ($n'^2=6-5i$); $h=0$.
(The dotted curves correspond to infinitely good conductivity.)

* *Ann. Physik*, Ser. 5, vol. 1 (1929), p. 734.

† In agreement with Mr. A. Sommerfeld the author plans to present a paper dealing with the physical problems mentioned in the above discussion, with the influence of the spherical form of the earth and with numerical details.

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